Multi-hadron observables from spectral functions

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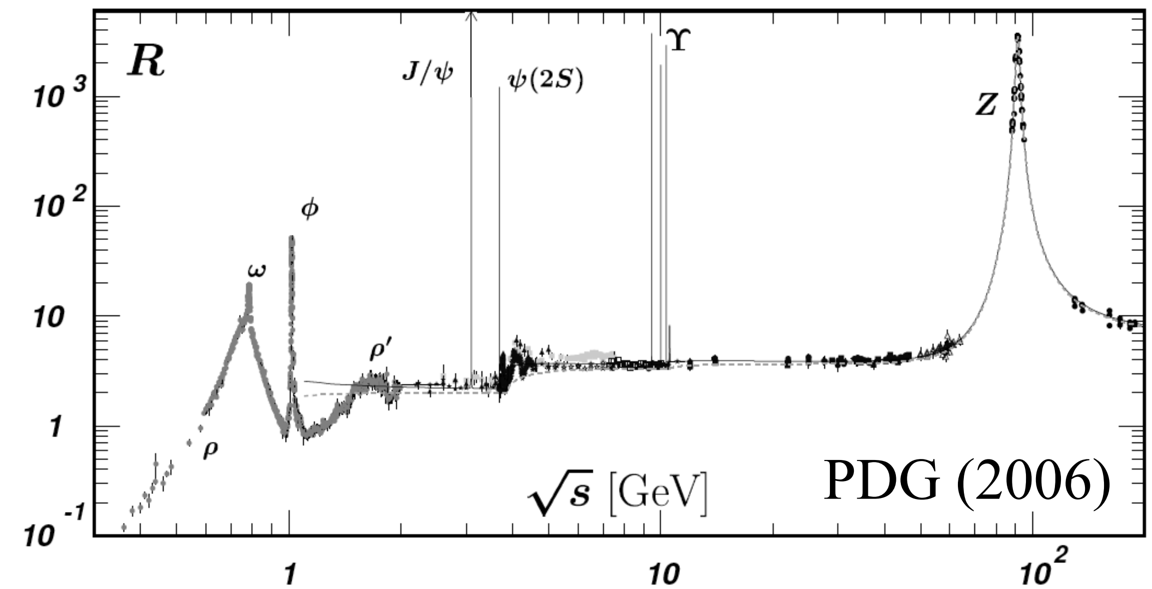
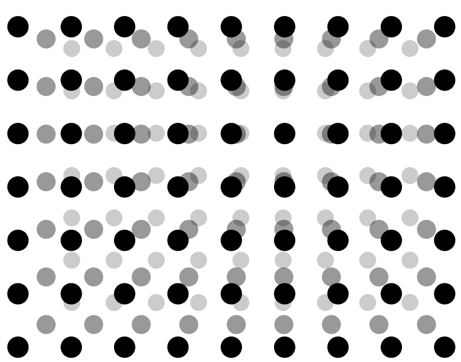
Invitation

- R-ratio based (g-2) determination

$$a^{\mathrm{HVP,LO}}(m_\mu) = \int_{4M_\pi^2}^{\infty} ds\, K(m_\mu, s)\, R(s)$$



- Integral can be Wick rotated to Euclidean signature… but ignore this



$$G(\tau) = \int_{4M_\pi^2}^{\infty} ds\, g(\tau, s)\, R(s) \qquad g(\tau, s) \propto \sqrt{s} e^{-\sqrt{s}\tau}$$

- Can we find a function $\mathcal{K}(m_\mu, \tau)$? such that…

$$\int_0^\infty d\tau\, \mathcal{K}(m_\mu, \tau)\, g(\tau, s) = K(m_\mu, s) \longrightarrow a^{\mathrm{HVP,LO}}(m_\mu) = \int_0^\infty d\tau\, \mathcal{K}(m_\mu, \tau)\, G(\tau)$$

Of course this defines the standard kernel

General idea

we want…

$$\widehat{\rho}(\bar{\omega}) \equiv \int_0^\infty d\omega\, \widehat{\delta}_\Delta(\bar{\omega}, \omega)\, \rho(\omega)$$

we have…

$$G(\tau) = \int d\omega\, e^{-\omega\tau}\, \rho(\omega)$$

we try to construct

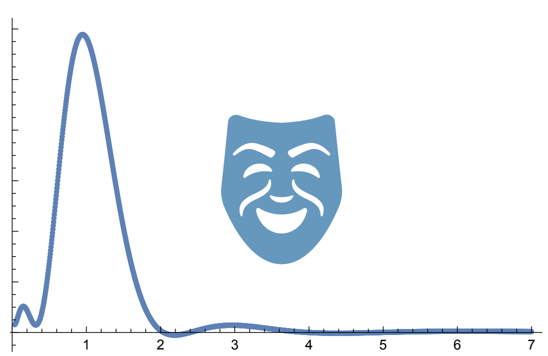
$$\int_0^\infty d\tau\, \mathcal{K}(\bar{\omega}, \tau)\, e^{-\omega\tau} = \widehat{\delta}_\Delta(\bar{\omega}, \omega)$$

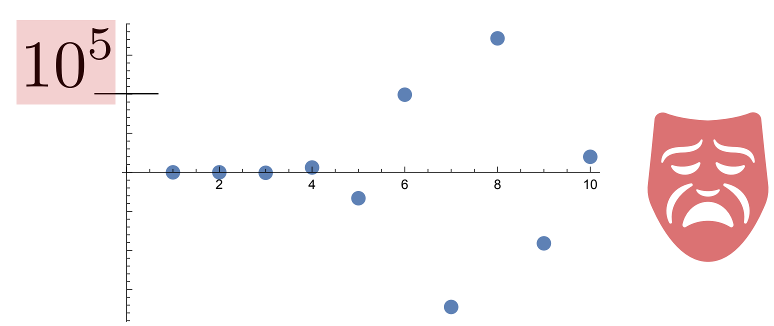
it then follows that

$$\widehat{\rho}(\bar{\omega}) = \int_0^\infty d\tau\, \mathcal{K}(\bar{\omega}, \tau)\, G(\tau)$$

some examples…

$\widehat{\delta}_\Delta(\bar{\omega}, \omega)$	$\mathcal{K}(\bar{\omega}, \tau)$
$e^{-\omega/\bar{\omega}}$	$\delta(\bar{\omega} - 1/\tau)$
$\dfrac{1}{\bar{\omega} - \omega}$	$e^{\bar{\omega}\tau}$

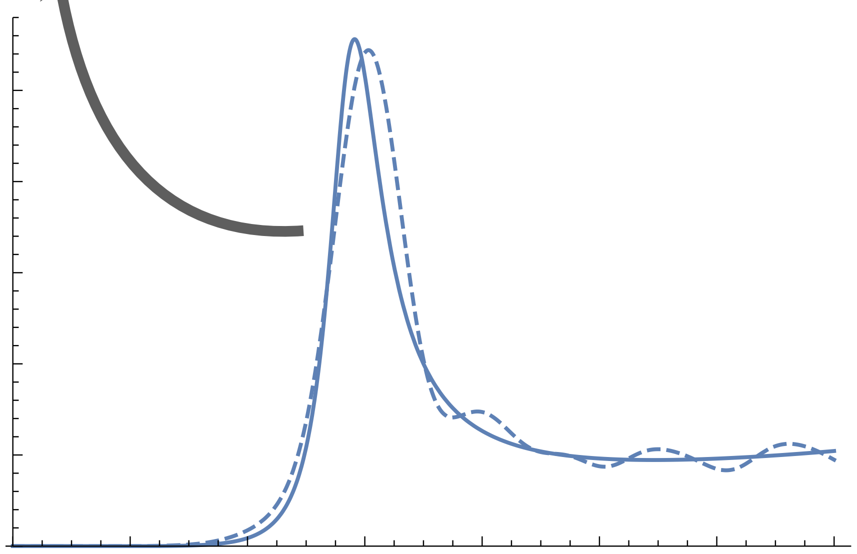


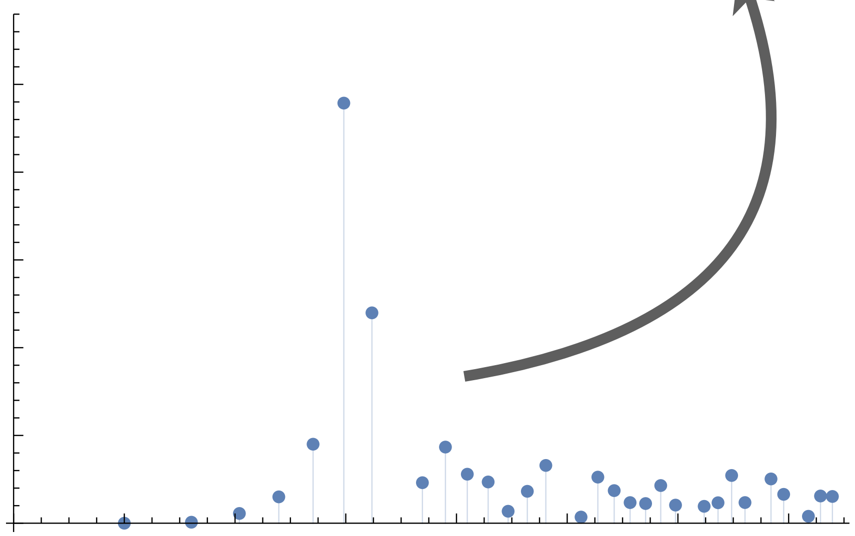


Role of the finite volume

$$\widehat{\rho}_L(\bar{\omega}) \equiv \int_0^\infty d\omega\, \widehat{\delta}_\Delta(\bar{\omega},\omega)\, \rho_L(\omega)$$

$$G_L(\tau) = \int d\omega\, e^{-\omega\tau}\, \rho_L(\omega)$$





- Any reconstructed spectral function that ≠ forest of deltas…

contains implicit smearing (or else $L \to \infty$)

We require…

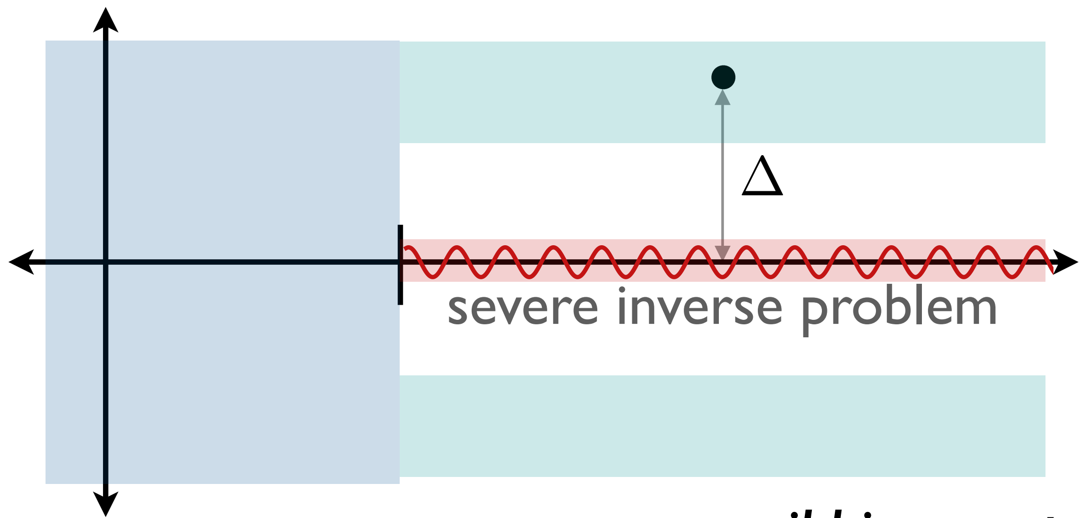
$$1/L \ll \Delta \ll \mu_{\text{physical}}$$

smearing function
covers many delta peaks

smearing does not overly
distort observable

MTH, Meyer, Robaina (2017)

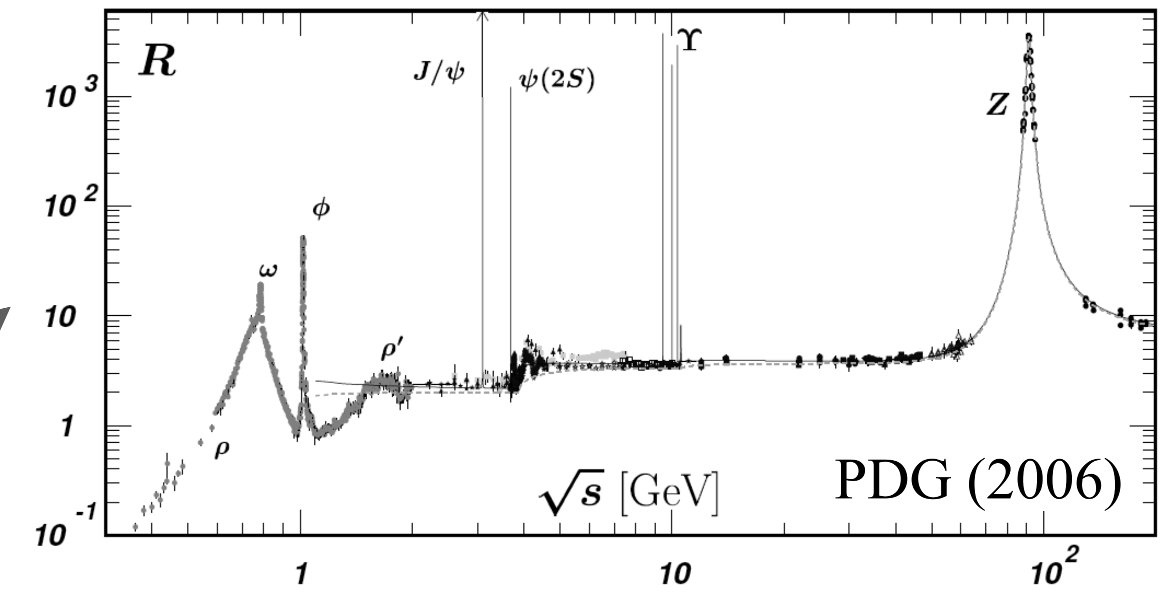
R ratio



no inverse problem

Hashimoto

mild inverse problem?



PHYSICAL REVIEW D VOLUME 13, NUMBER 7 1 APRIL 1976

Smearing method in the quark model*

E. C. Poggio, H. R. Quinn,† and S. Weinberg

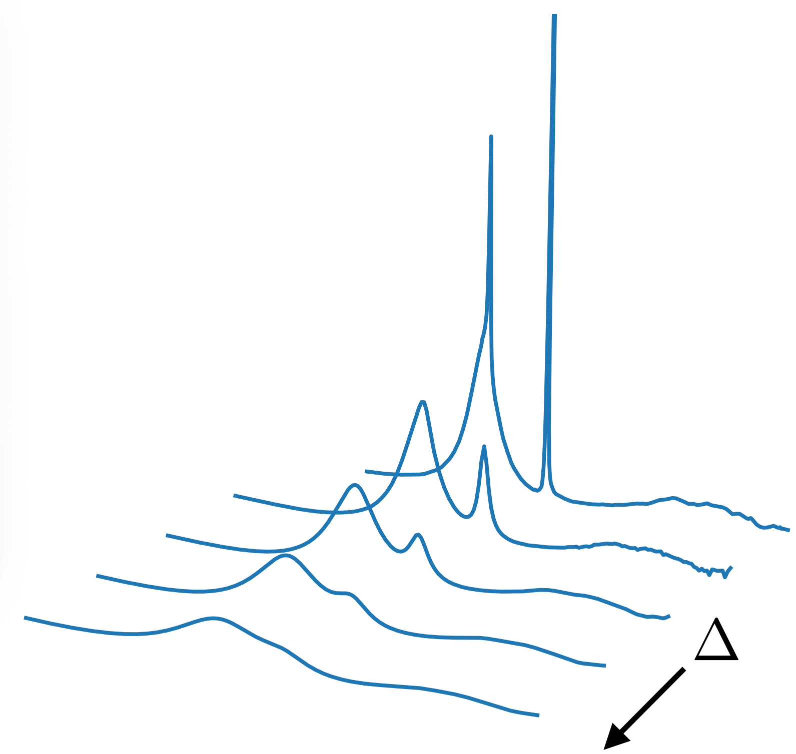
Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138

(Received 8 December 1975)

We propose that perturbation theory in a gauge field theory of quarks and gluons can be used to calculate physical mass-shell processes, provided that both the data and the theory are smeared over a suitable energy range. This procedure is explored in detail for the process of electron-positron annihilation into hadrons. We show that a smearing range of 3 GeV^2 in the squared center-of-mass energy should be adequate to allow the use of lowest-order perturbation theory. The smeared data are compared with theory for a variety of models. It appears to be difficult to fit the present data with theory, unless there is a new charged lepton or quark with mass in the region 2 to 3 GeV.

$$\widehat{R}_\Delta(s) = \frac{\Delta}{\pi}\int_0^\infty ds' \, \frac{R(s')}{(s'-s)^2+\Delta^2}$$

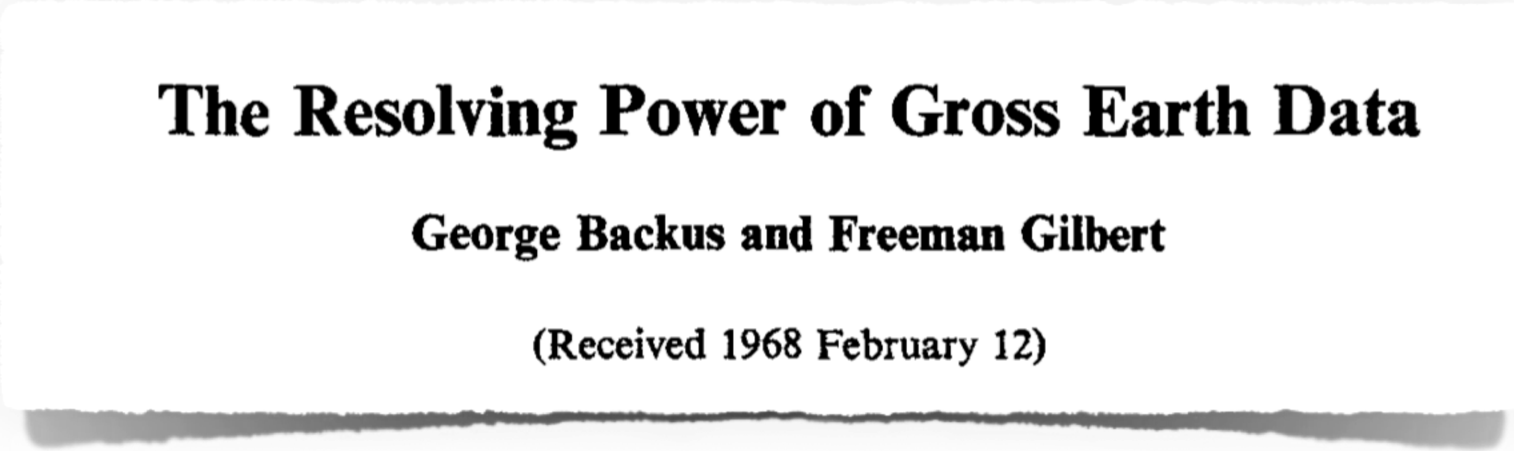
into the complex plane!



courtesy of M. Bruno
(using F. Jegerlehner's ***alphaQED***)

Backus-Gilbert

- Developed by geophysicists Backus and Gilbert in 1967



The Resolving Power of Gross Earth Data

George Backus and Freeman Gilbert

(Received 1968 February 12)

- ***Linear, model-independent*** → smeared solution with a ***known resolution function***

$$\sum_\tau \mathcal{K}(\bar{\omega},\tau)G(\tau) = \sum_\tau \mathcal{K}(\bar{\omega},\tau)\int d\omega\, e^{-\omega\tau}\,\rho(\omega)$$

$$= \int d\omega\,\Big[\sum_\tau \mathcal{K}(\bar{\omega},\tau)\,e^{-\omega\tau}\Big]\,\rho(\omega)$$

$$= \int d\omega\,\widehat{\delta}_\Delta(\bar{\omega},\omega)\,\rho(\omega)$$

BG optimizes *K*

δ is exactly known

- The correlation matrix is used to stabilize the inverse

- The quality of the data determines the ***resolution***

Backus-Gilbert (details)

- Un-stabilized inverse

Without uncertainties, minimize the functional…

$$A[\mathcal{K}] \equiv \int_0^\infty d\omega\,(\bar\omega-\omega)^2\,\widehat{\delta}_\Delta(\bar\omega,\omega)^2 = \int_0^\infty d\omega\,(\bar\omega-\omega)^2\left[\sum_\tau \mathcal{K}_\tau e^{-\omega\tau}\right]^2$$

with unit area constraint on δ

- Stabilized inverse

With uncertainties, instead minimize…

$$W_\lambda[\mathcal{K}] \equiv (1-\lambda)A[\mathcal{K}] + \lambda\frac{\mathcal{K}\cdot\mathrm{COV}\cdot\mathcal{K}}{\mathcal{N}}$$

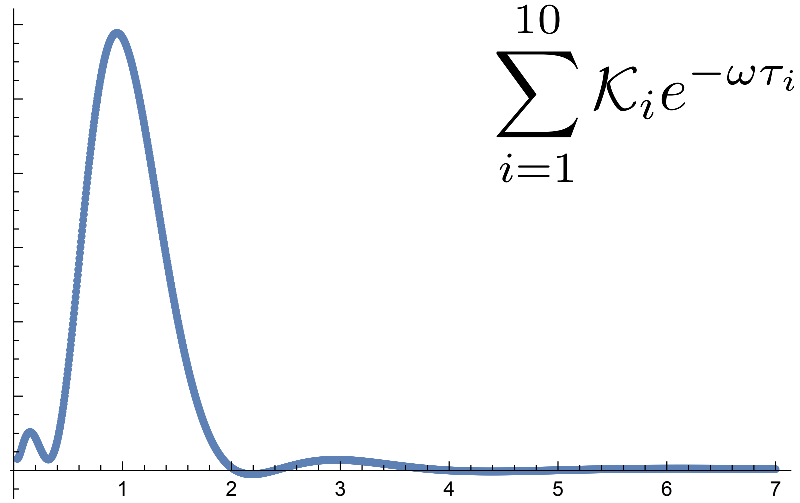
wants oscillating K

wants well-behaved K

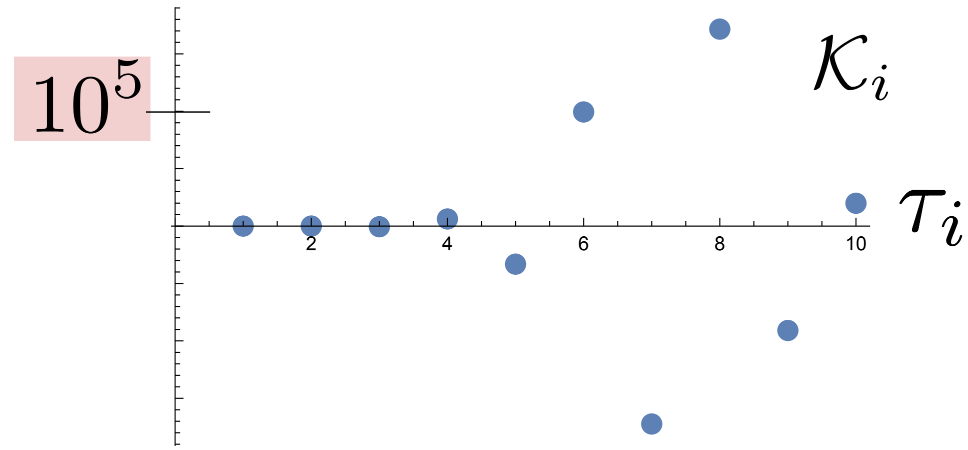
λ parametrizes the tradeoff between *final resolution* and *final* uncertainty

Backus-Gilbert example

- Un-stabilized inverse

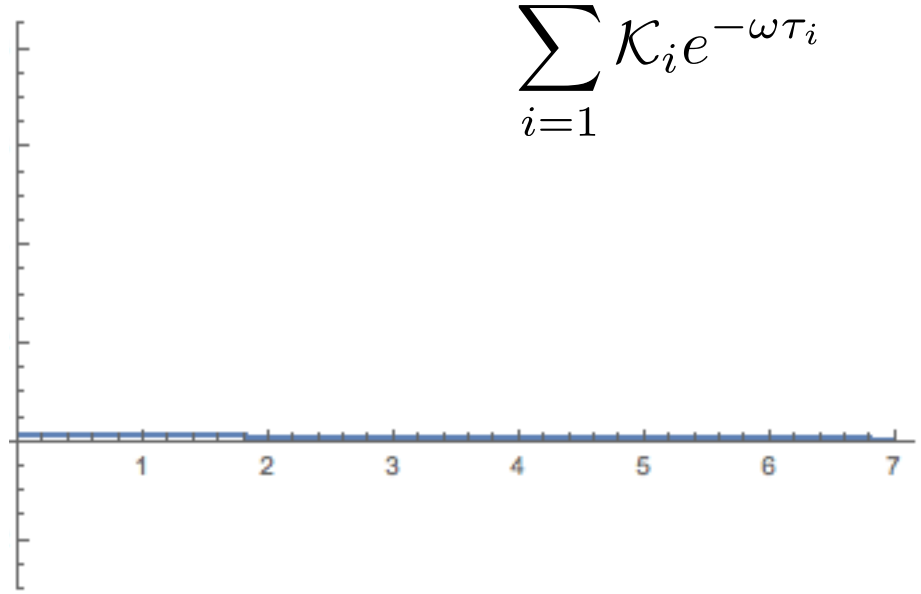


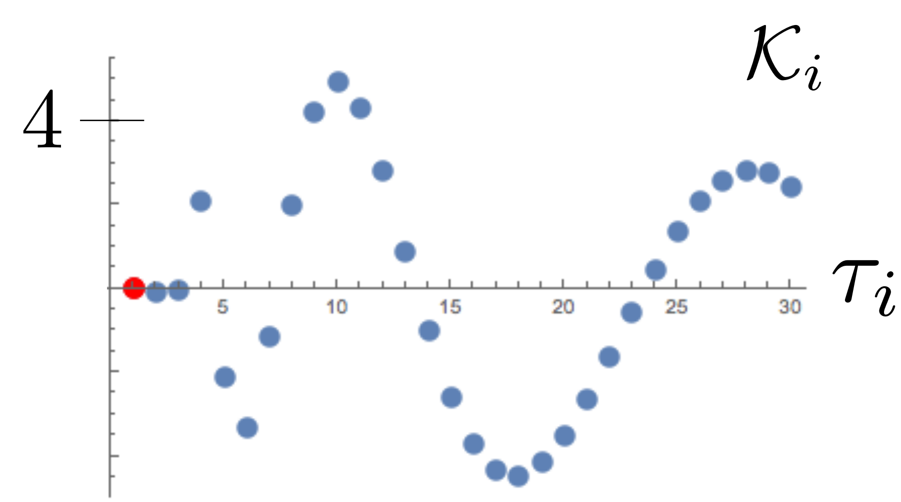
$$\sum_{i=1}^{10} \mathcal{K}_i e^{-\omega\tau_i}$$



- Stabilized inverse

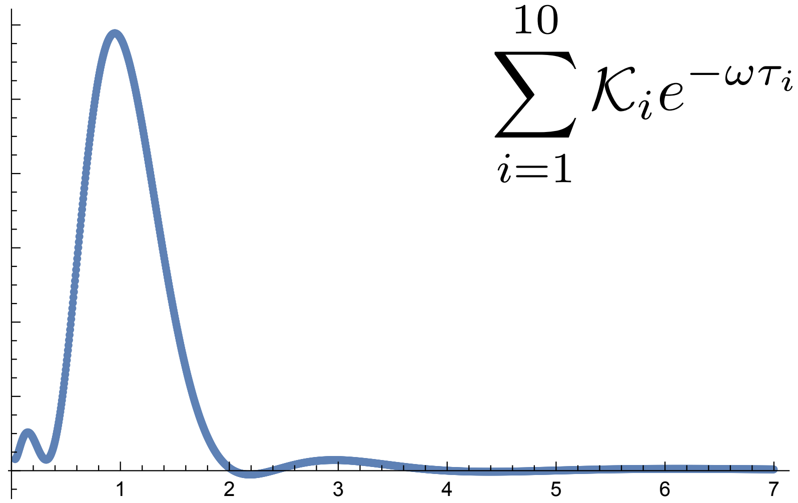
$$\sum_{i=1}^{\bullet} \mathcal{K}_i e^{-\omega\tau_i}$$





Backus-Gilbert example

- Un-stabilized inverse



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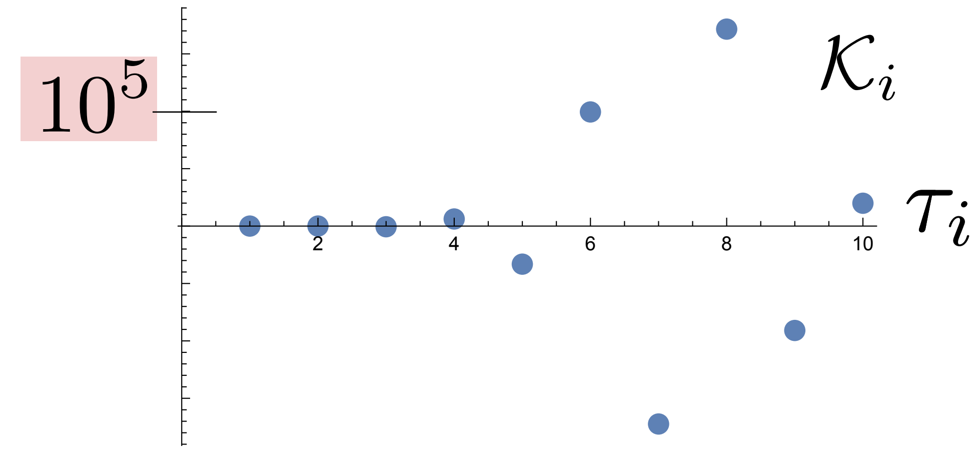
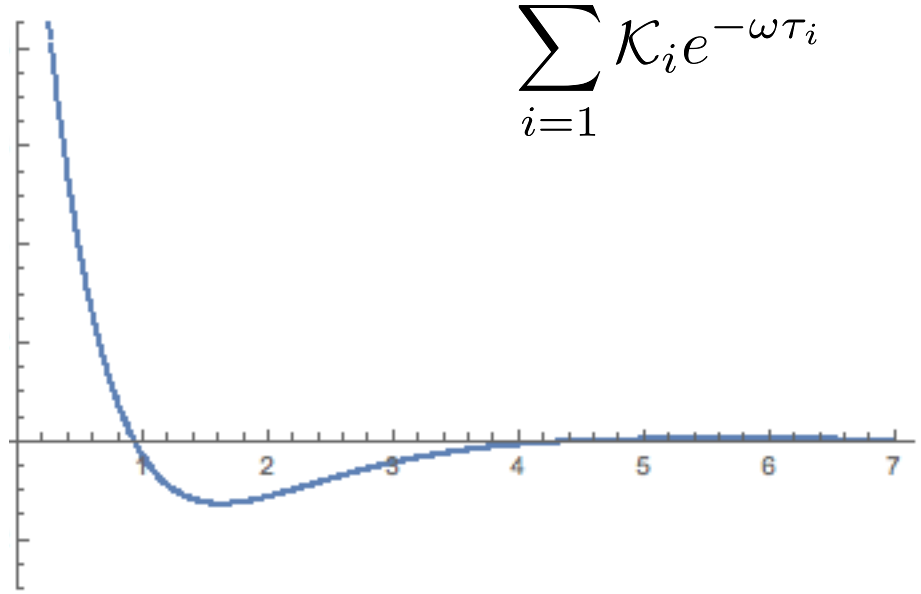
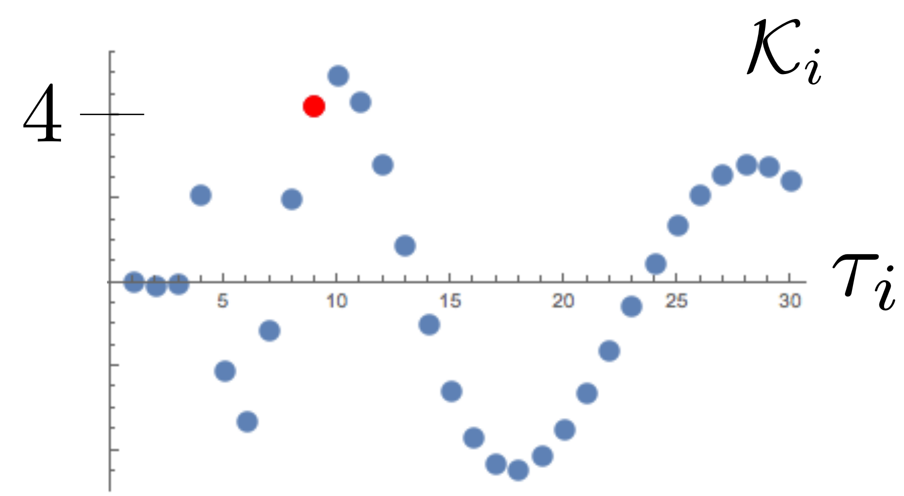


- Stabilized inverse

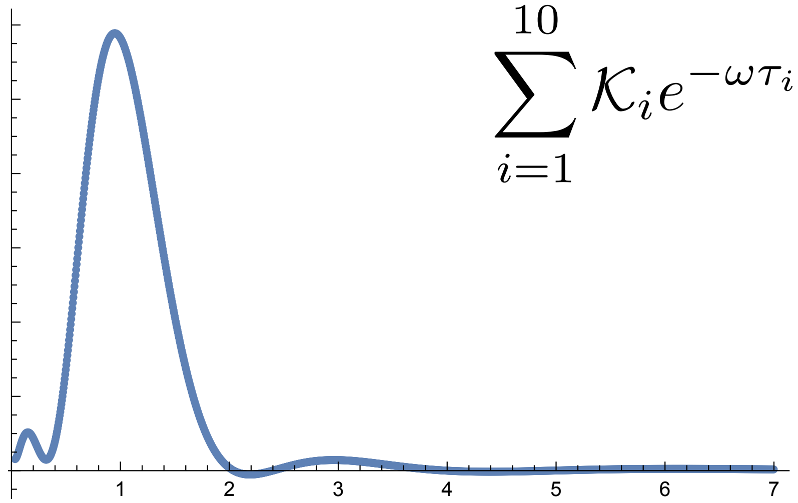


$$\sum_{i=1} \mathcal{K}_i e^{-\omega \tau_i}$$



Backus-Gilbert example

- Un-stabilized inverse



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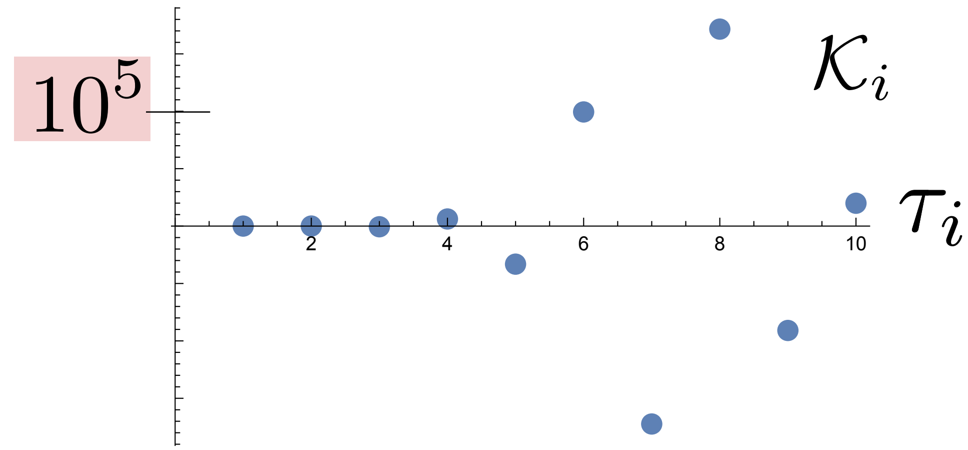
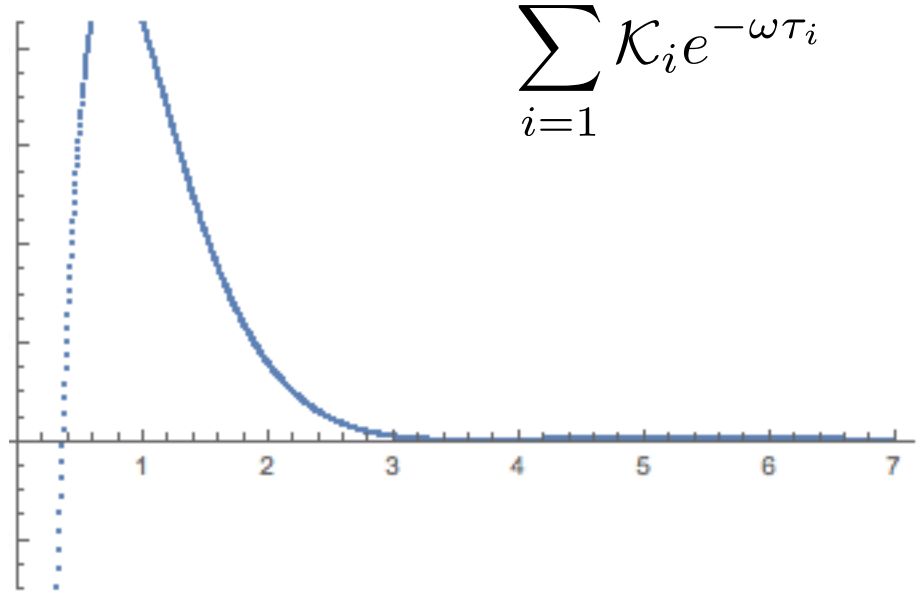


- Stabilized inverse



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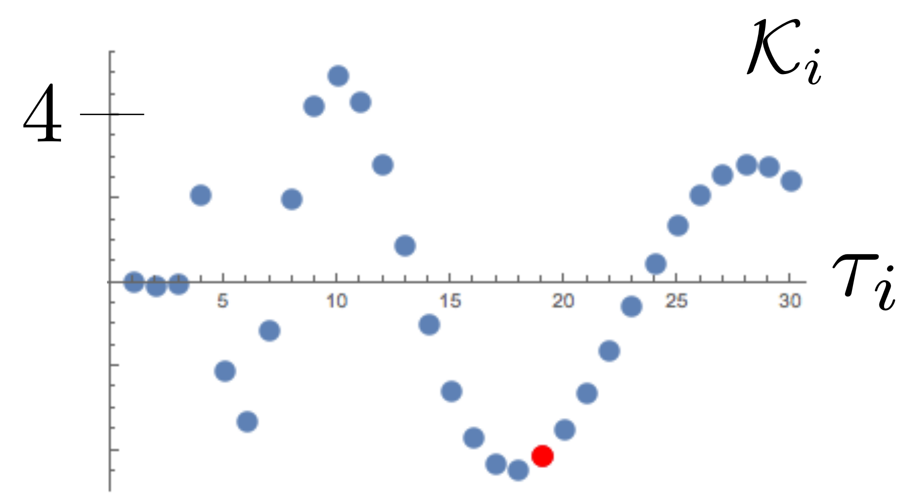
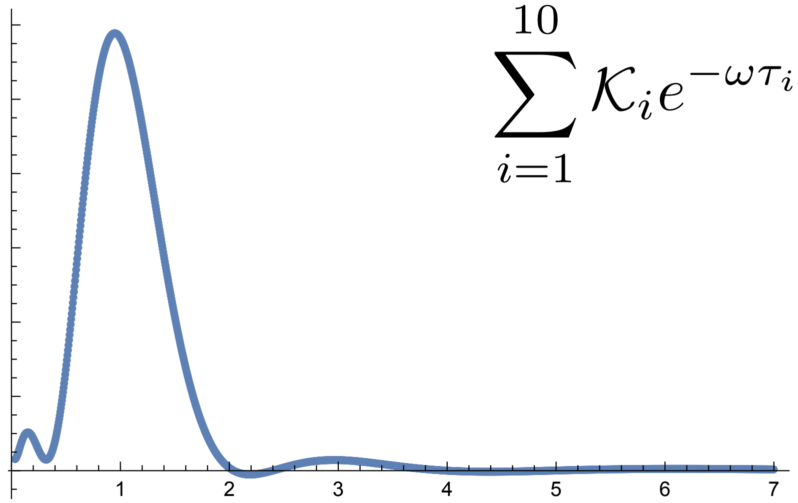


Backus-Gilbert example

- Un-stabilized inverse



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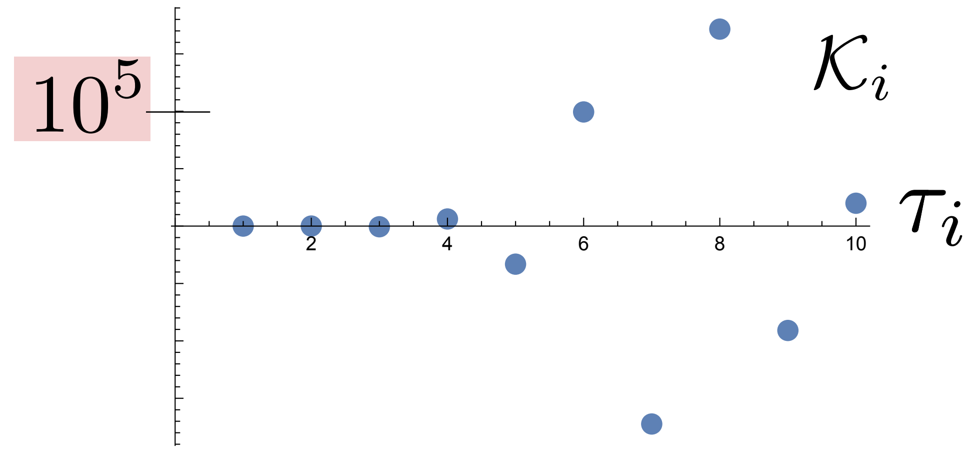
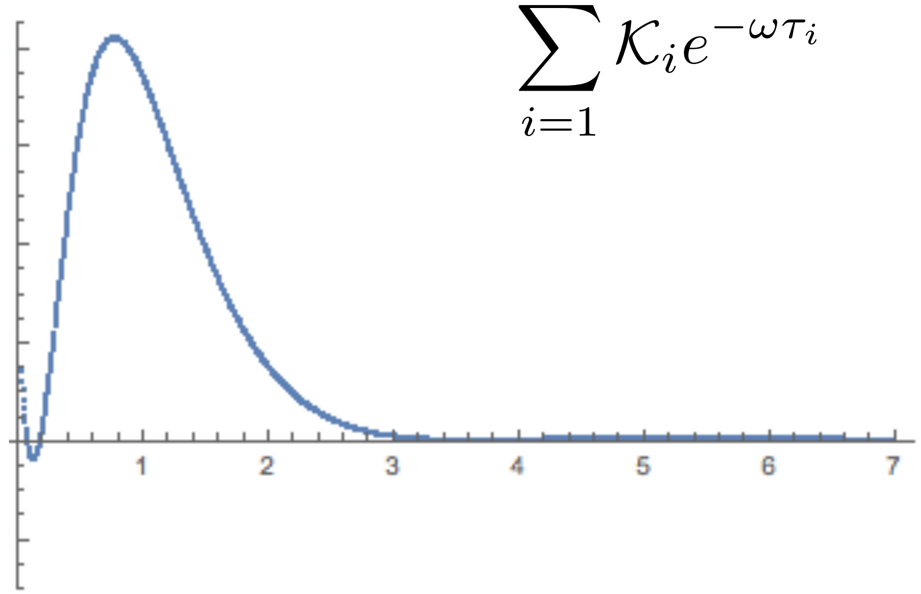
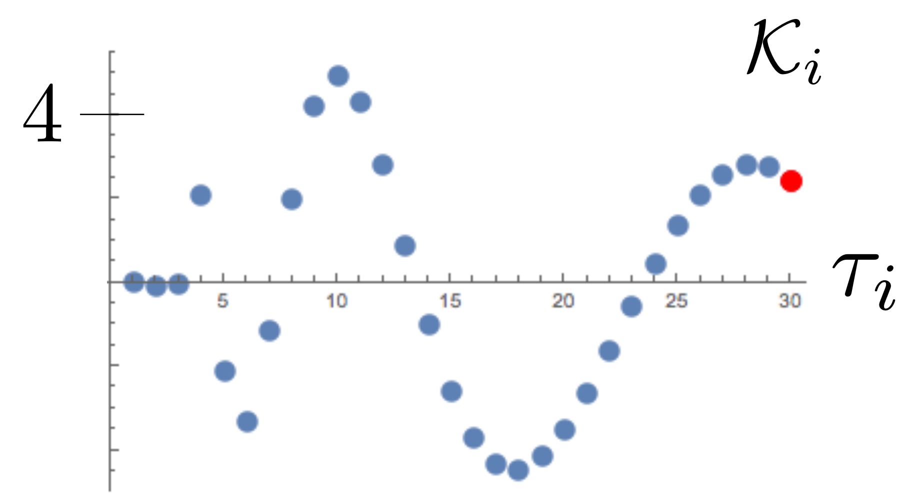


- Stabilized inverse



$$\sum_{i=1}^{\bullet} \mathcal{K}_i e^{-\omega \tau_i}$$



Extended Backus Gilbert

- Important generalization from ***Martin Hansen, Lupo, Tantalo***
- Addressed two limitations…

Shape and width of δ depends on data quality

No clear way to set λ

$$W_\lambda[\mathcal{K}] \equiv (1-\lambda)A[\mathcal{K}] + \lambda \frac{\mathcal{K}\cdot \mathrm{COV}\cdot \mathcal{K}}{\mathcal{N}}$$

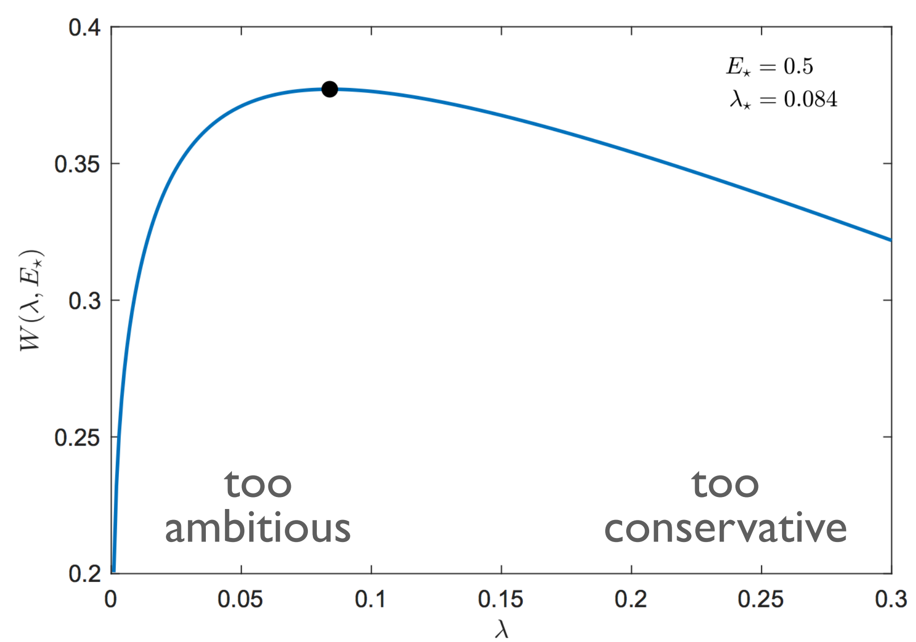
Change minimizing functional

$$A[\mathcal{K}] \equiv \int_0^\infty d\omega\, (\bar\omega-\omega)^2\, \widehat{\delta}_\Delta(\bar\omega,\omega)^2$$

$$\widetilde{A}[\mathcal{K}] \equiv \int_0^\infty d\omega \left[\widehat{\delta}_\Delta^{\,\mathrm{target}}(\bar\omega,\omega) - \widehat{\delta}_\Delta(\bar\omega,\omega)\right]^2$$

Extremize *W w.r.t.* λ

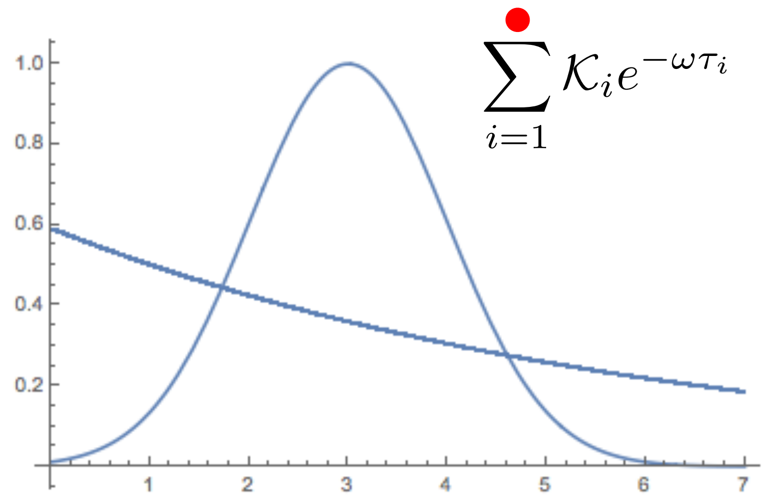
$$W_\lambda[\mathcal{K}_{\min}(\lambda)]$$

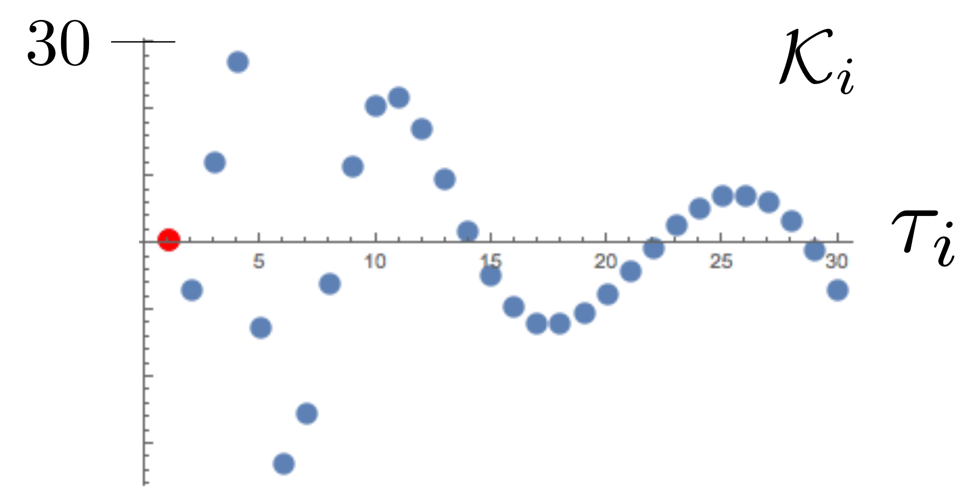


Hansen, Lupo, Tantalo (2019)

Extended Backus Gilbert example

- e.g. target a Gaussian





Extended Backus Gilbert example

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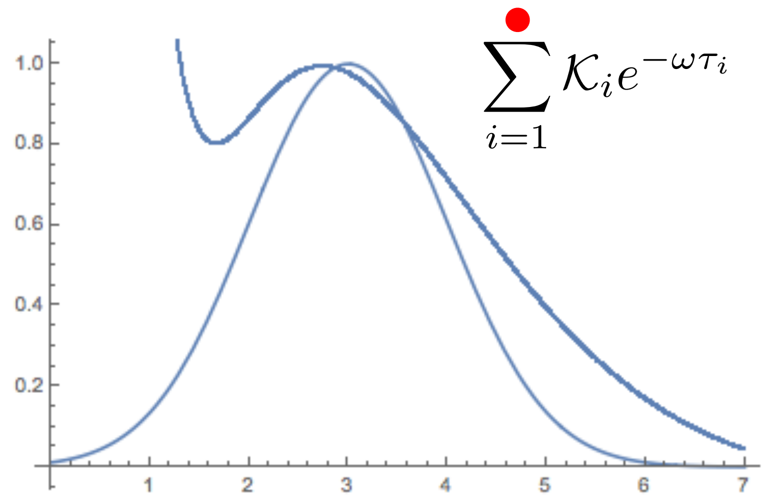

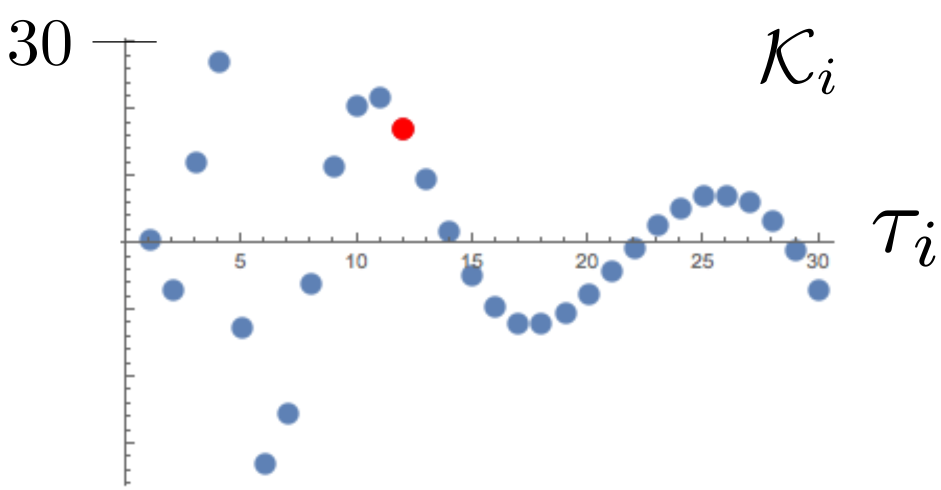

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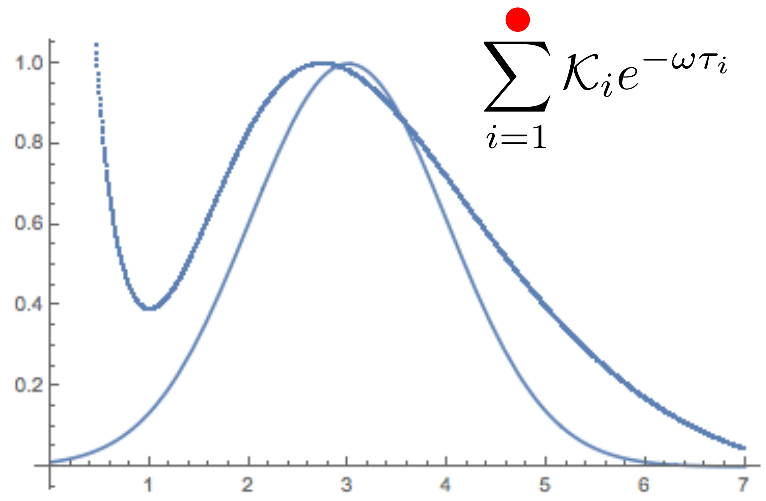
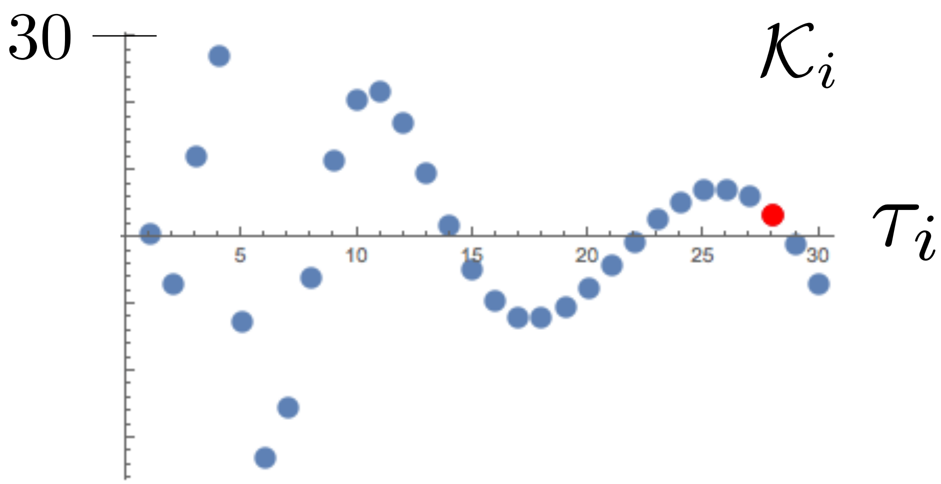
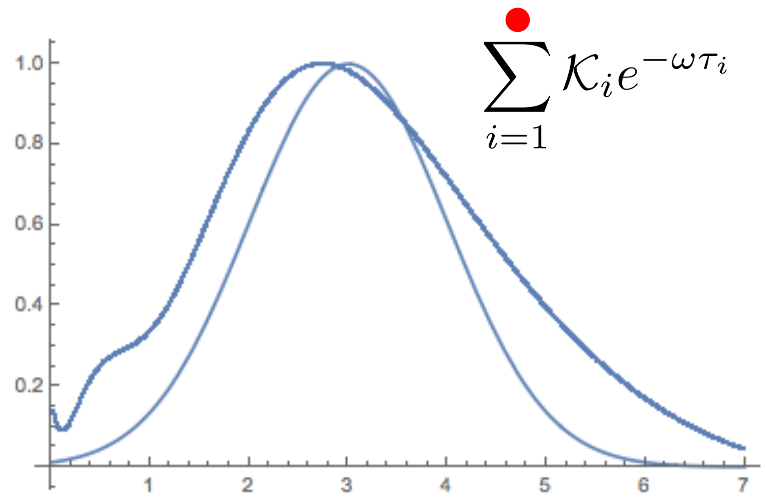
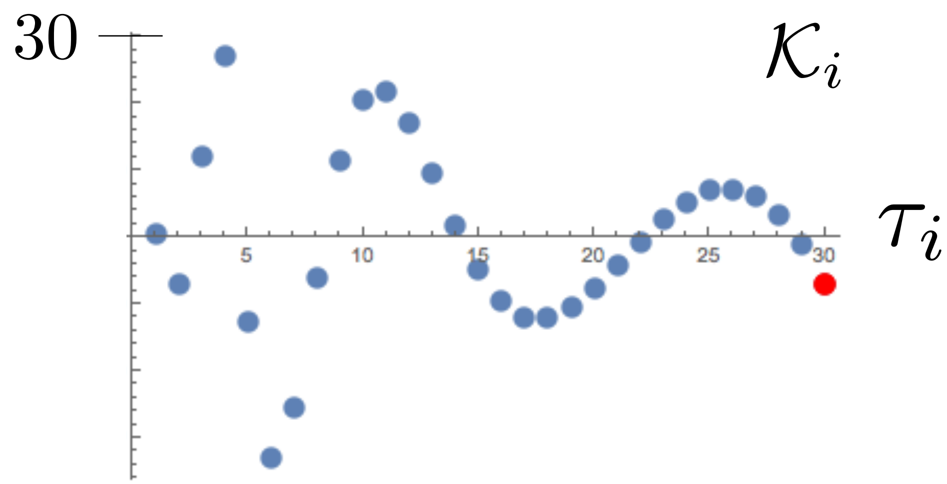




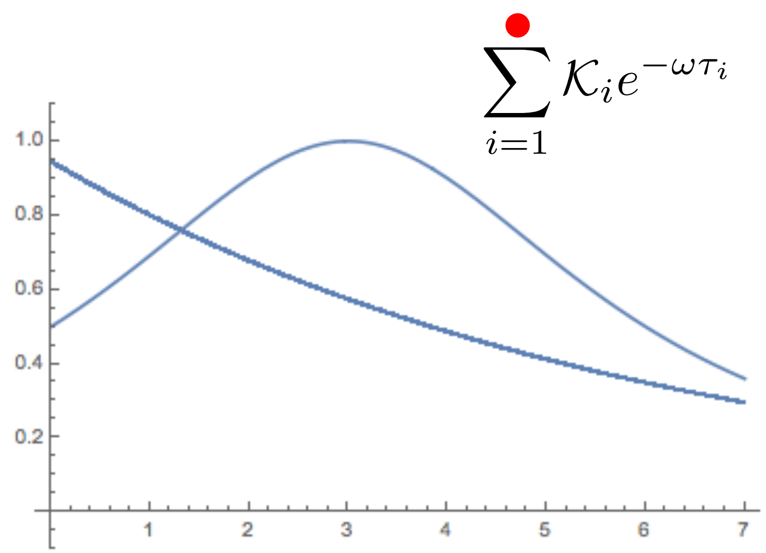
Extended Backus Gilbert example

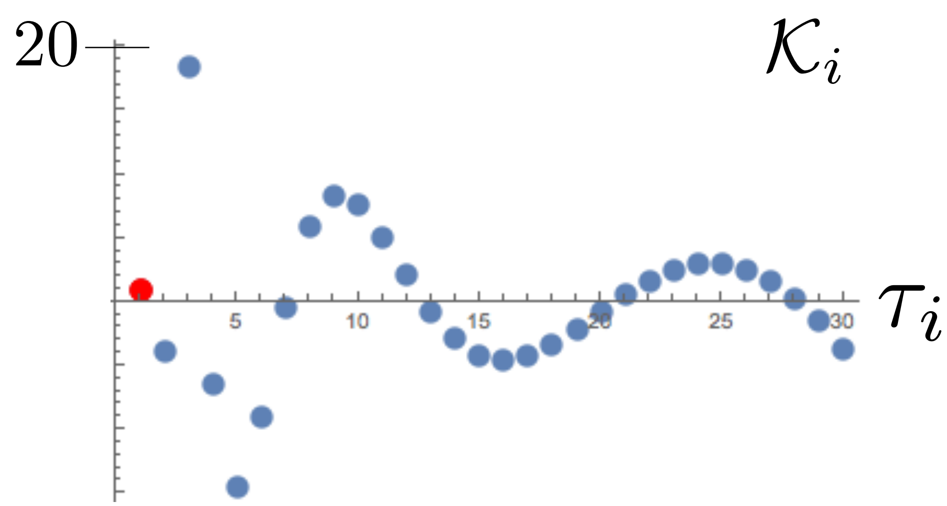
- e.g. target a Gaussian





- … or a Breit Wigner





Extended Backus Gilbert example

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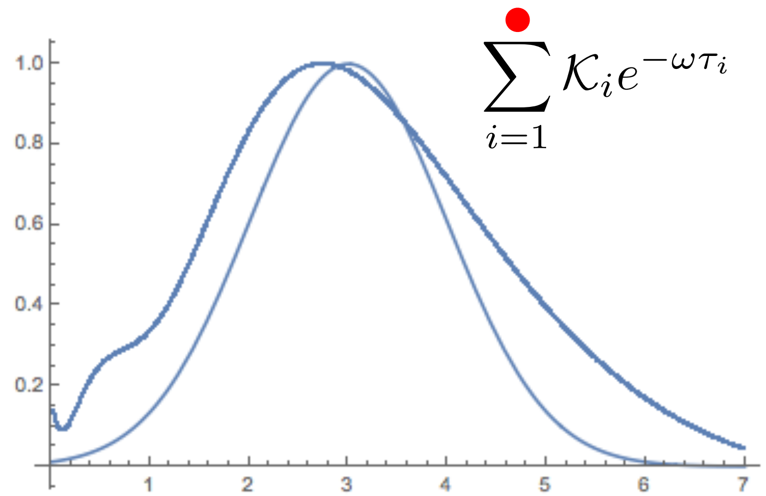

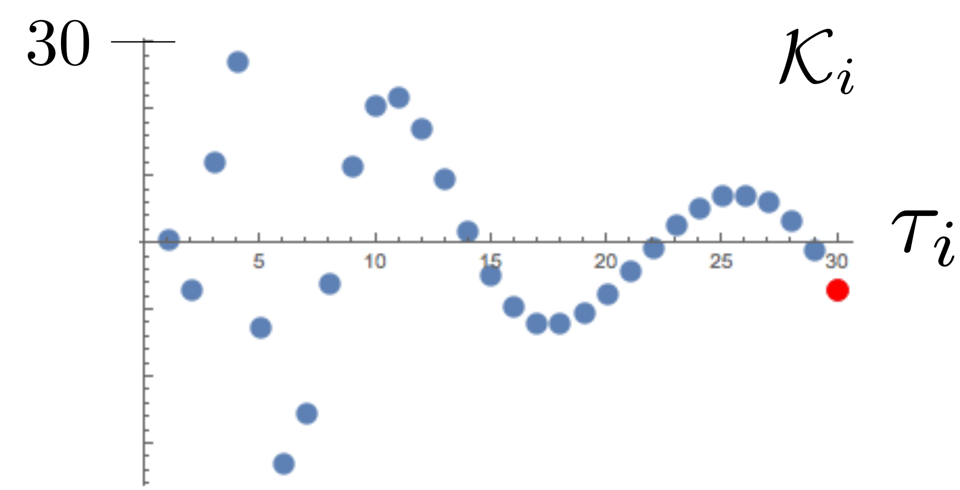

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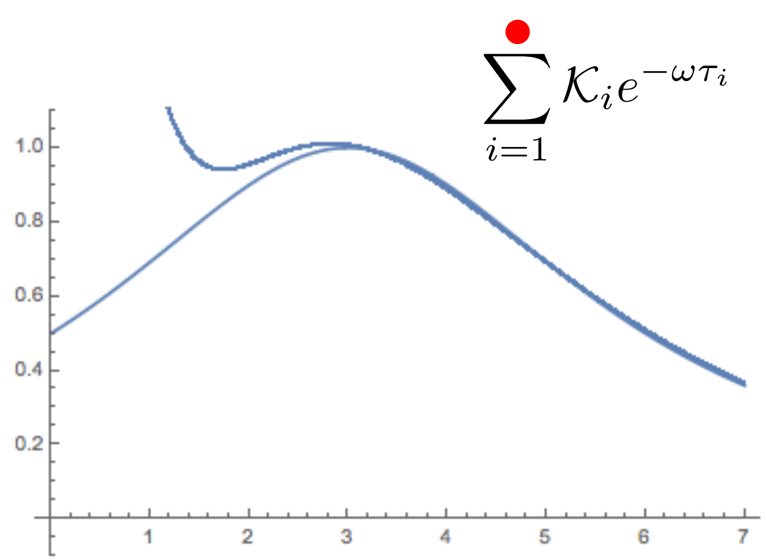

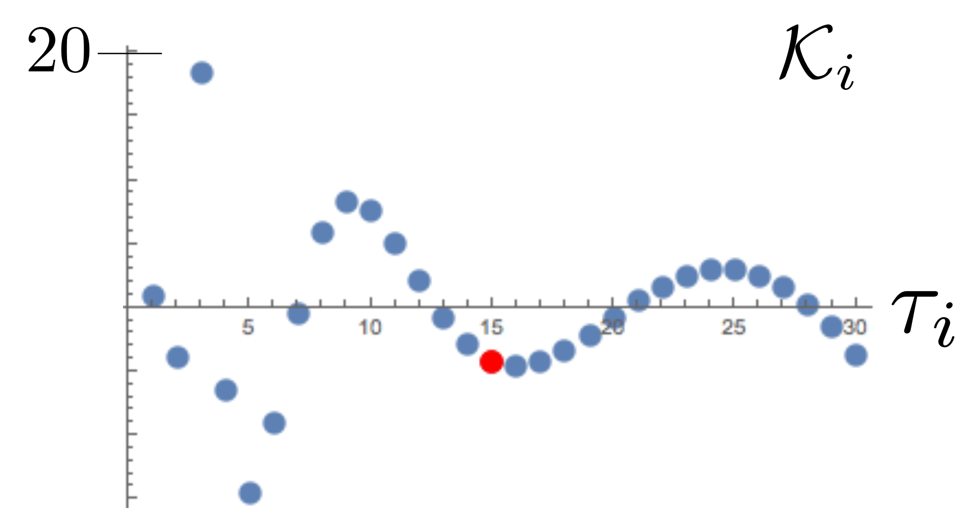

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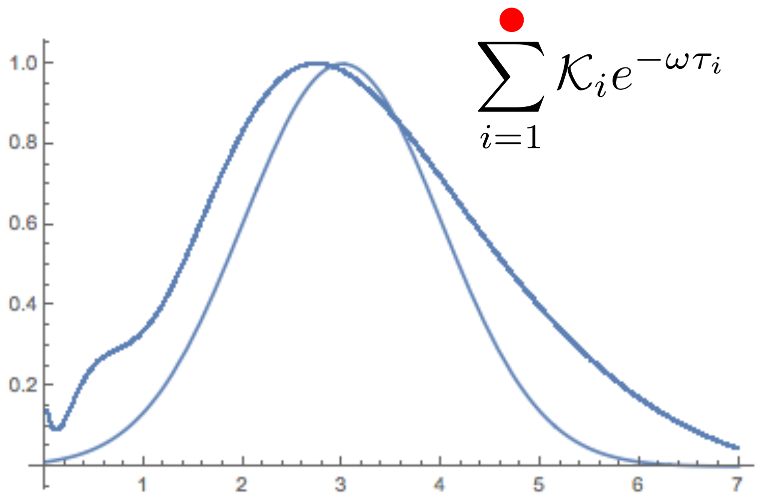

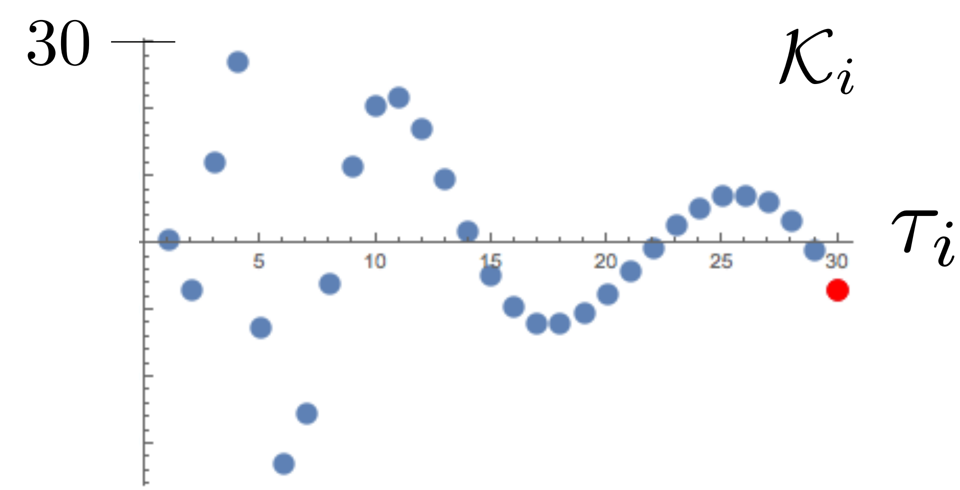

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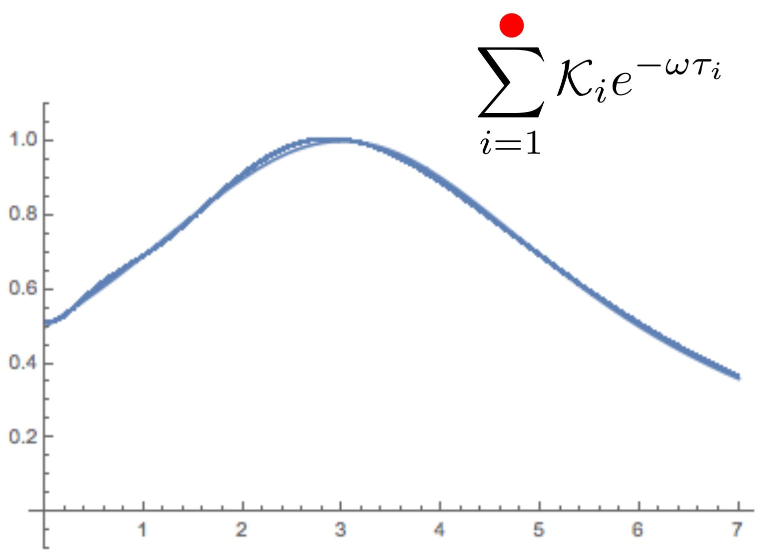
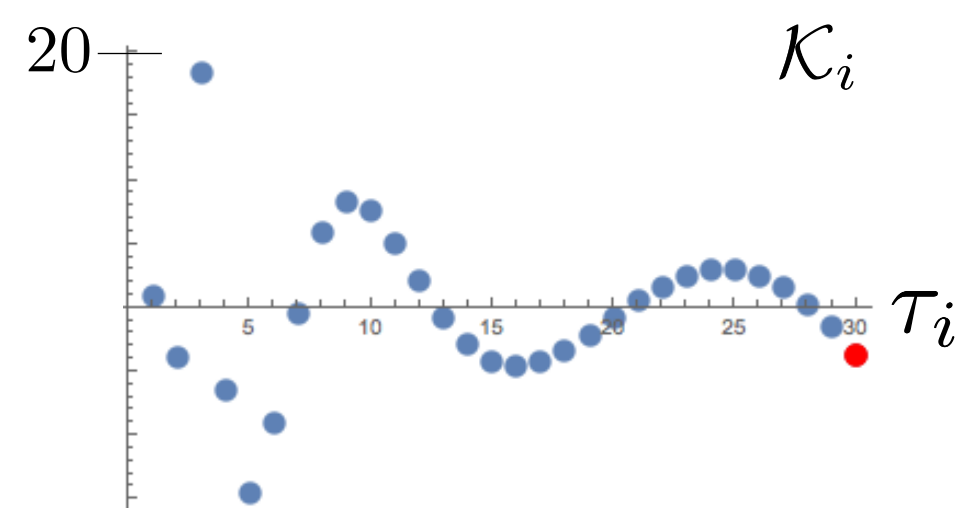
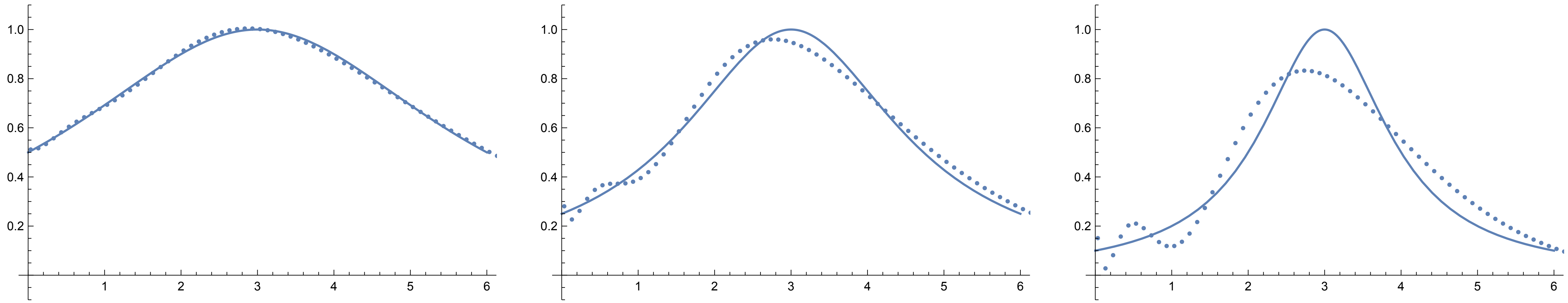




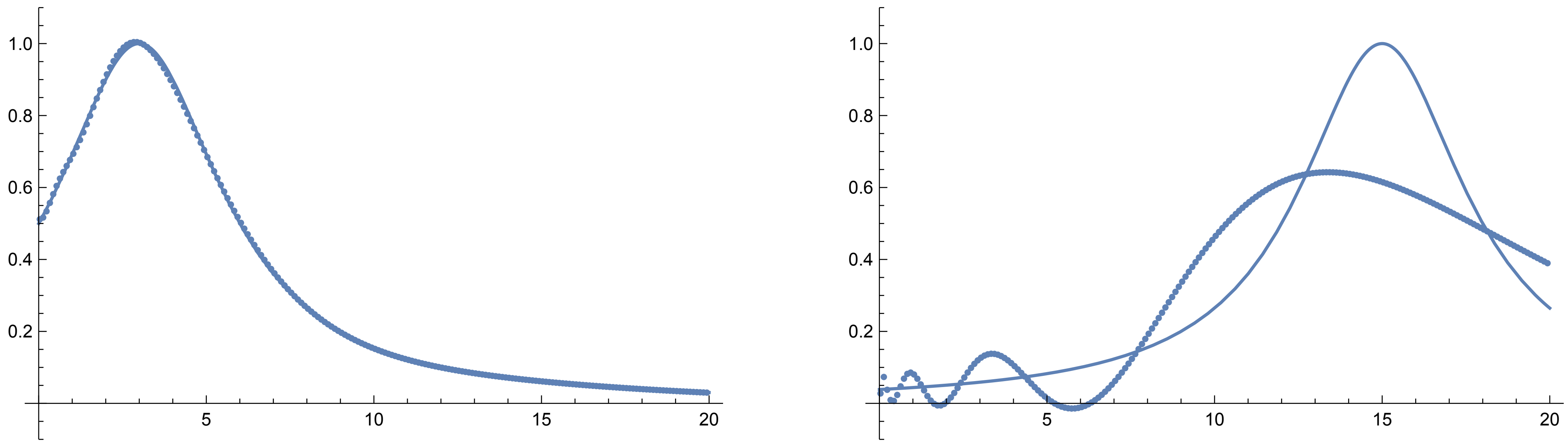
Conservation of evil

- Method will always fail if one tries to…

make the resolution function too narrow



move the resolution too high in energy



Hybrid approach

- We do have a robust method to solve the inverse problem... ***GEVP***
- Suggests a separation of low-lying states

$$G^{\mathrm{GEVP}}(\tau) = \sum_{n}^{E_n < 6M_\pi} c_n(L) e^{-E_n(L)\tau} \qquad G(\tau) = \sum_{n}^{\infty} c_n(L) e^{-E_n(L)\tau} = \int_{2M_\pi}^{\infty} d\omega\, \rho(\omega)\, e^{-\omega\tau}$$

$$G^{\mathrm{sub}}(\tau) \equiv G(\tau) - G^{\mathrm{GEVP}}(\tau) = \sum_{n}^{E_n > 6M_\pi} c_n(L) e^{-E_n(L)\tau}$$

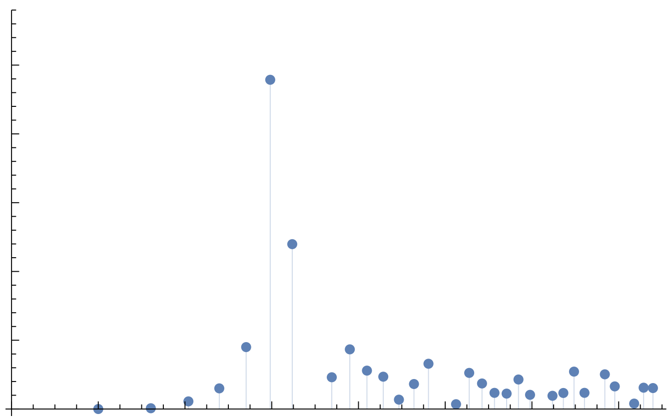
$$= \int_{6M_\pi}^{\infty} d\omega\, \rho(\omega)\, e^{-\omega\tau}$$

- Suppose we want $\widehat{\rho}(\omega = 8M_\pi)$

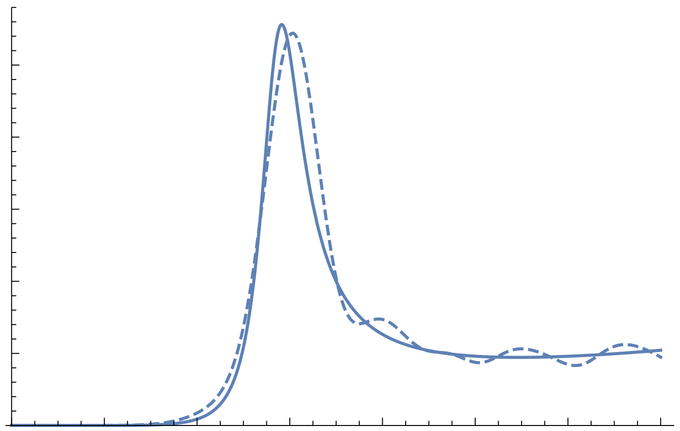
resolution from		resolution from		resolution from
$G^{\mathrm{sub}}(\tau)$	$\approx$	$G(\tau)$	$\gg$	$G(\tau)$
at $\omega = 8M_\pi$		at $\omega = 4M_\pi$		at $\omega = 8M_\pi$

Intermediate summary

- Cannot solve the inverse problem, we can get $\widehat{\rho}_{L,\Delta}(\bar{\omega}) \equiv \int_0^\infty d\omega\, \widehat{\delta}_\Delta(\bar{\omega},\omega)\, \rho_L(\omega)$
- Smearing is needed anyway to ***suppress volume effects***

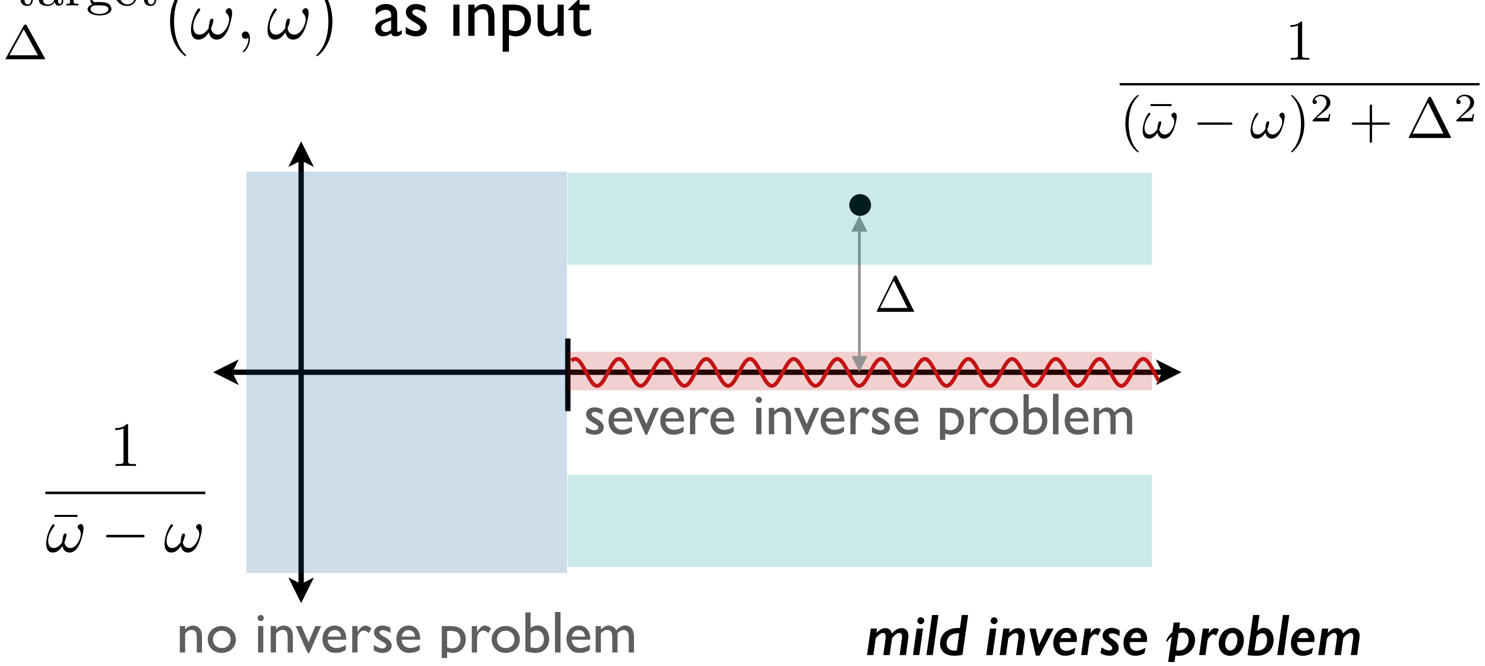


$$1/L \ll \Delta \ll \mu_{\text{physical}}$$



- Generalized Backus-Gilbert takes $\widehat{\delta}^{\,\text{target}}_\Delta(\bar{\omega},\omega)$ as input

defines a continuum of
inverse-problem severity

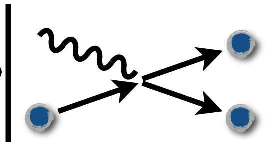
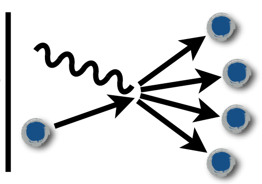
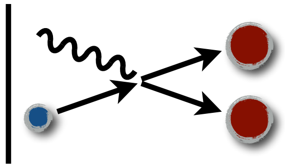


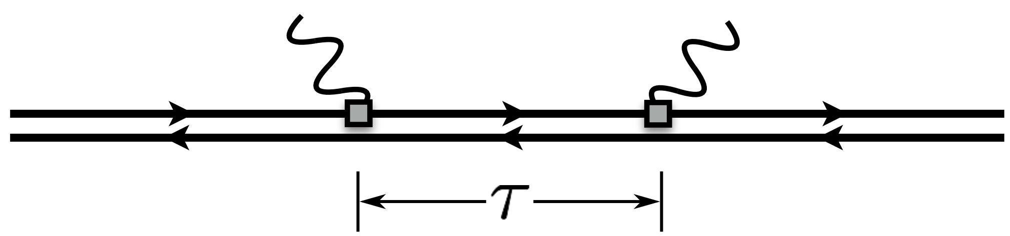
- Remove low lying states → start BG when states are dense

Total rate based applications

- Hadronic tensor, heavy flavor lifetimes

$$W_{\mu\nu}(p,q) \equiv \int d^4x \; e^{iqx} \langle \pi, p | \mathcal{J}_\mu^\dagger(x) \mathcal{J}_\nu(0) | \pi, p \rangle$$

$\propto \int d\Phi \Big|$  $\Big|^2 + \int d\Phi \Big|$  $\Big|^2 + \int d\Phi \Big|$  $\Big|^2 + \cdots$



$$= \int_0^\infty d\omega \, e^{-\omega \tau} \, W_{\mu\nu}(p,q)_{\omega = p^0 + q^0}$$

$$W_{\mu\nu} = \lim_{\Delta \to 0} \lim_{L \to \infty} \widehat{W}_{\mu\nu;\Delta,L}$$

- What about ***scattering*** and ***transition amplitudes?***

MTH, Meyer, Robaina (2017)

Amplitudes from spectral functions

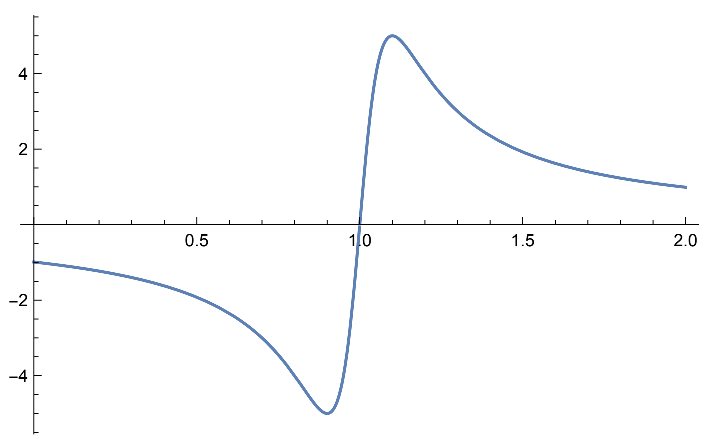
- First define the Euclidean correlator

$$G_L(\tau) = \langle \pi_{\boldsymbol{p}_4} | \pi(\tau_3, \boldsymbol{p}_3) \pi(0) | \pi_{\boldsymbol{p}_1} \rangle_L = \int_0^\infty dE_3 \, \rho(E_3) e^{-E_3 \tau_3}$$

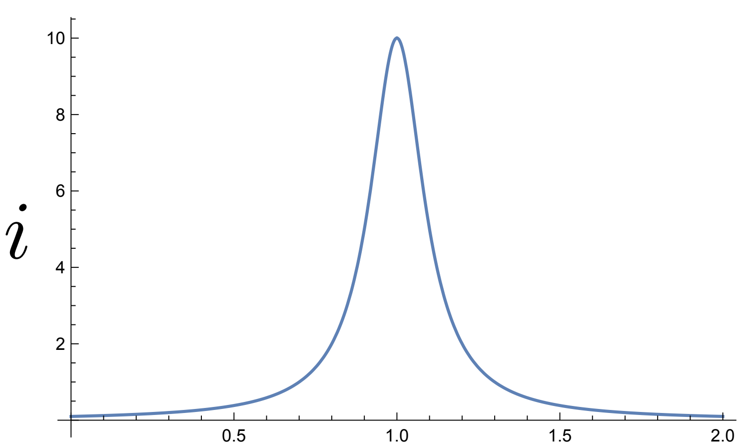
and the smeared spectral function

$$\widehat{\rho}^{L,\epsilon}_{\boldsymbol{p}_4 \boldsymbol{p}_1}(q_3) = \int_0^\infty dE_3 \, \frac{1}{q_3^0 - E_3 + i\epsilon} \, \rho(E_3) = \int_0^\infty dE_3 \, \widehat{\delta}_\epsilon(q_3^0, E_2) \, \rho(E_3)$$

$$\frac{1}{q_3^0 - E_3 + i\epsilon} =$$



$$-i$$



Bulava, MTH (2019)

Amplitudes from spectral functions

- First define the Euclidean correlator

$$G_L(\tau) = \langle \pi_{\boldsymbol{p}_4} | \pi(\tau_3, \boldsymbol{p}_3) \pi(0) | \pi_{\boldsymbol{p}_1} \rangle_L = \int_0^\infty dE_3 \, \rho(E_3) e^{-E_3 \tau_3}$$

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- Next project *on shell* at finite ϵ

$$\mathcal{M}_{\mathrm{c}}^{L,\epsilon}(p_4 p_3 | p_2 p_1) \equiv \frac{2E(\boldsymbol{p}_3)}{Z^{1/2}(\boldsymbol{p}_3)} \frac{2E(\boldsymbol{p}_2)}{Z^{1/2}(\boldsymbol{p}_2)} \epsilon^2 \, \widehat{\rho}^{L,\epsilon}_{\boldsymbol{p}_4 \boldsymbol{p}_1}(E(\boldsymbol{p}_3), \boldsymbol{p}_3)$$

- Finally project out the ***scattering amplitude***

$$\mathcal{M}_{\mathrm{c}}(p_4 p_3 | p_2 p_1) = \lim_{\epsilon \to 0^+} \lim_{L \to \infty} \mathcal{M}_{\mathrm{c}}^{L,\epsilon}(p_4 p_3 | p_2 p_1)$$

Bulava, MTH (2019)

Some comments

- Derivation based in modified **LSZ** + *signature-independence* of ρ(E)
- Holds when LSZ holds

$$\langle m, \text{out}|n, \text{in}\rangle \qquad \langle m, \text{out}|\mathcal{J}(0)|n, \text{in}\rangle$$

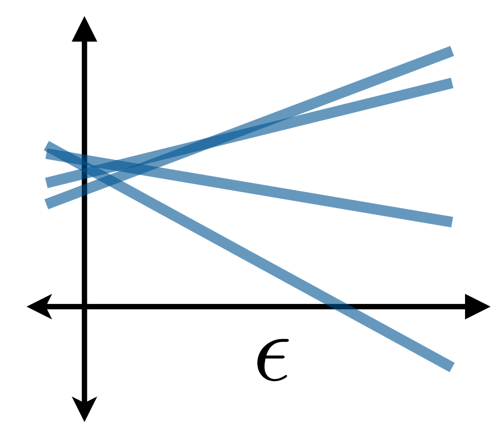
- Very challenging… but systematic

  for some (unknown) volume + correlator quality, we can get $D \to \pi\pi, K\overline{K}$

- Some nice features…

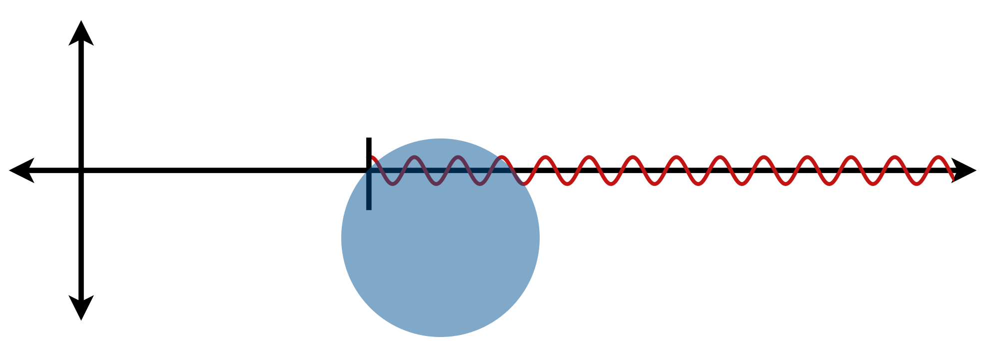
  GEVP-like operator freedom

$$G_L^{[ab]}(\tau) = \langle \pi_{\boldsymbol{p}_4}|\pi^a(\tau_3,\boldsymbol{p}_3)\pi^b(0)|\pi_{\boldsymbol{p}_1}\rangle_L \longrightarrow \mathcal{M}^{[ab],\epsilon,L}$$



Finite range of analyticity in ϵ

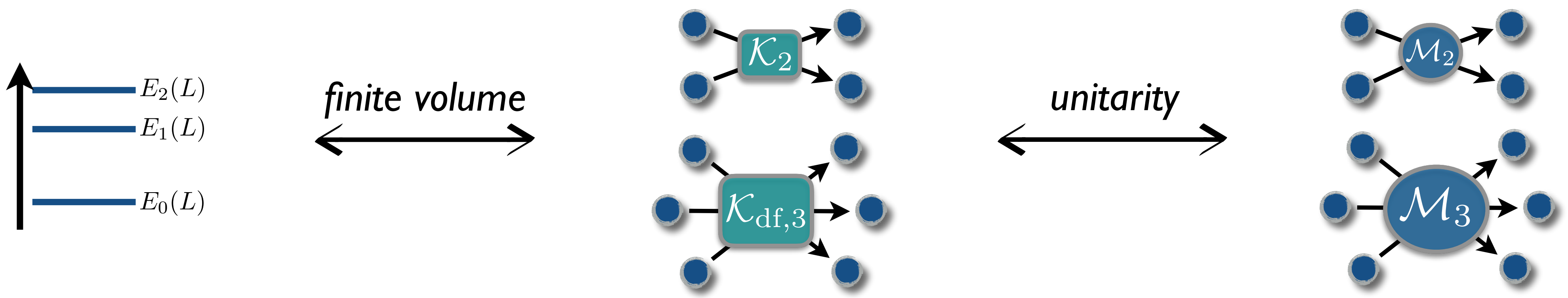
Uses overlaps



Bulava, MTH (2019)

Interlude…

- This is an alternative to finite-volume methods



- Proven very powerful and are here to stay

- Spectral function methods will be most competitive in multi-channel regime

- Biggest impact may be lower-precision long distance effects
  e.g. QED corrections of D decays with multi-hadron intermediate states

- Spectral function methods use ***matrix elements*** and ***energies***

Huang, Yang (1958) • Lüscher (1986, 1991) • Rummukainen, Gottlieb (1995)
Kim, Sachrajda, Sharpe (2005) • Christ, Kim, Yamazaki (2005) • He, Feng, Liu (2005)
Leskovec, Prelovsek (2012) • Bernard *et. al.* (2012) • MTH, Sharpe (2012) • Briceño, Davoudi (2012)
Li, Liu (2013) • Briceño (2014) • Briceño, Davoudi (2015) • Mai, Döring • Hansen, Sharpe

Perturbative study…

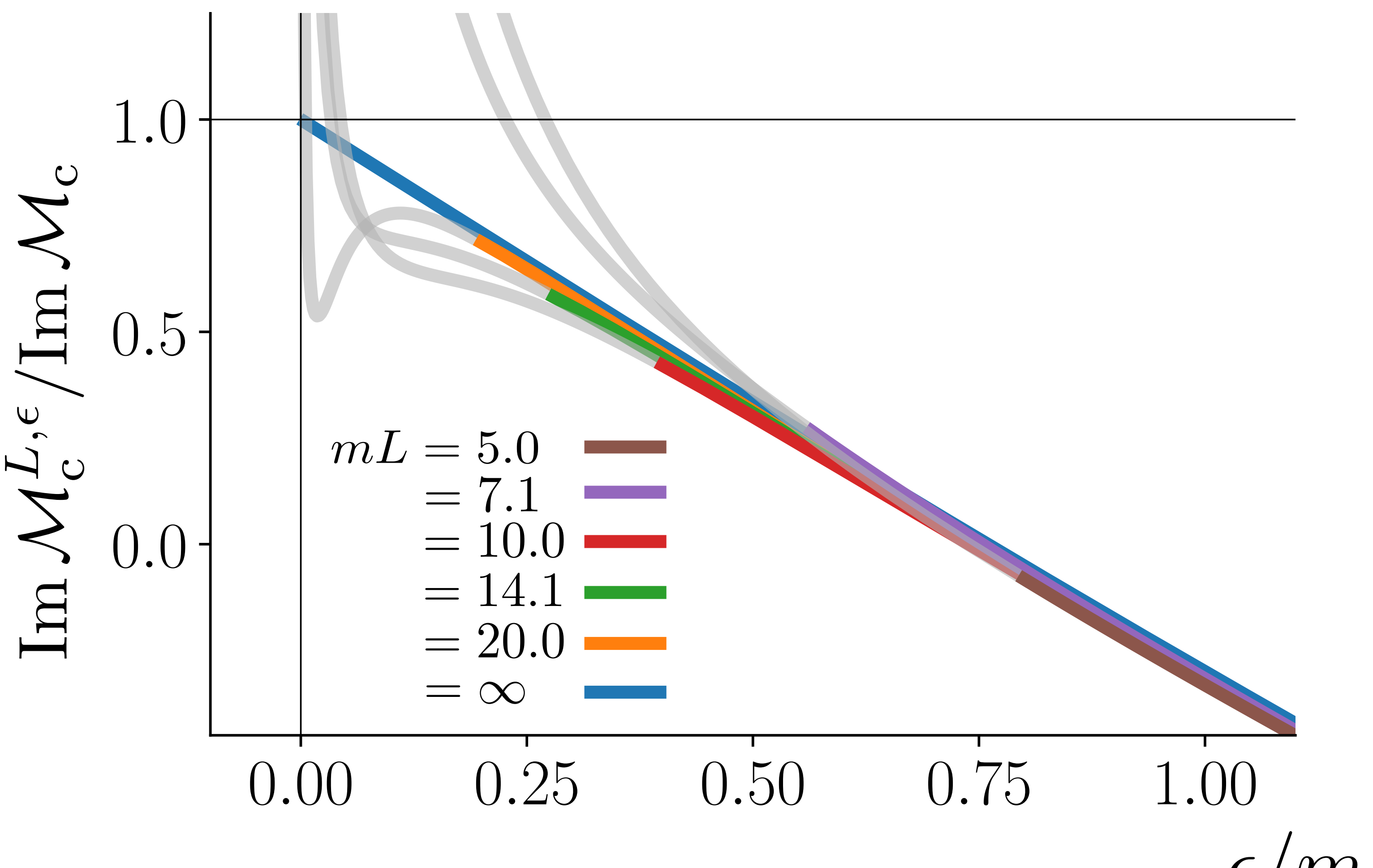
Calculate in PT

$$G_L(\tau) = \langle \pi_{\boldsymbol{p}_4} | \pi(\tau_3, \boldsymbol{p}_3) \pi(0) | \pi_{\boldsymbol{p}_1} \rangle_L$$

Convert to this

$$\mathcal{M}_{\rm c}^{L,\epsilon}(p_4 p_3 | p_2 p_1)$$

$$\mathrm{Im}\,\mathcal{M}_{\rm c}^{L,\epsilon}(p_4 p_3 | p_2 p_1) = \frac{\lambda^2}{2} \frac{1}{L^3} \sum_{\boldsymbol{k}'}^{\Lambda} \frac{1}{(2E(\boldsymbol{k}'))^2}\, \mathrm{Im} \left\{ \frac{1}{(E_{\rm cm} - 2E(\boldsymbol{k}') + i\epsilon)} \left[1 - \frac{\epsilon^2}{4E(\boldsymbol{k}')^2} - \frac{\epsilon(\epsilon + 2iE(\boldsymbol{k}'))}{E_{\rm cm} E(\boldsymbol{k}')} \right] \right\}$$



Bulava, MTH (2019)

Perturbative study…

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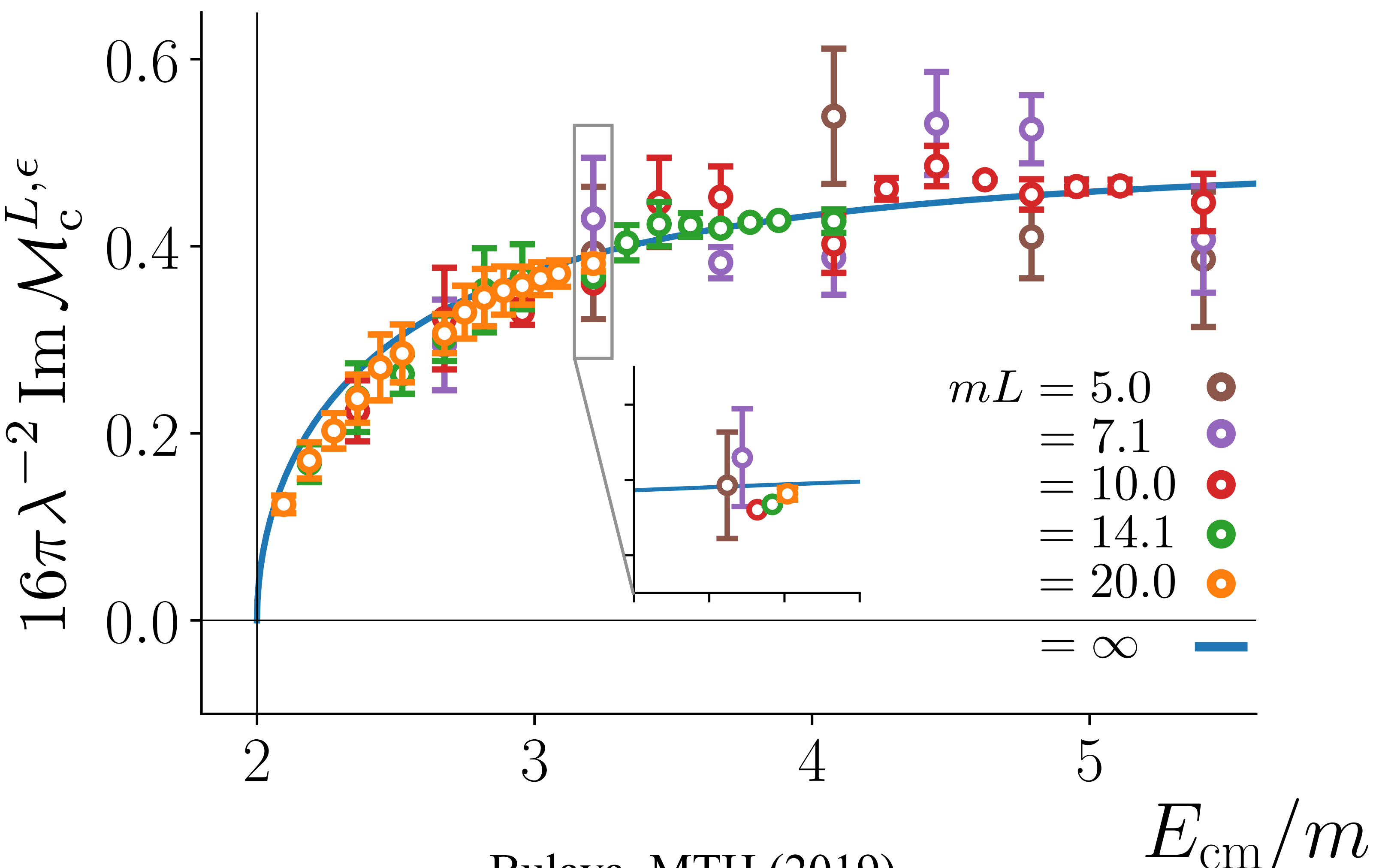


Bulava, MTH (2019)

Connection to Maiani-Testa

- Maiani and Testa considered correlators of the form

$$G_{[\boldsymbol{p}]}(\tau) = \langle \pi_{-\boldsymbol{p}} | \pi_{\boldsymbol{p}}(\tau) J(0) | 0 \rangle$$

- And showed two key points

$$G_{[\mathbf{0}]}(\tau) = \frac{\sqrt{Z_\pi}}{2M_\pi} e^{-M_\pi \tau} f(4M_\pi^2) \left[1 - a_{\pi\pi} \sqrt{\frac{m_\pi}{\pi \tau}} + O(\tau^{-3/2}) \right]$$

$G_{[\boldsymbol{p} \neq \mathbf{0}]}(\tau)$ is plagued by un-physical (operator dependent) contributions

- Recent work with M. Bruno connects this story to spectral function amplitudes

Variations on the Maiani-Testa approach and the inverse problem

M. Bruno[a] and M. T. Hansen[b]

arXiv: 2012.11488

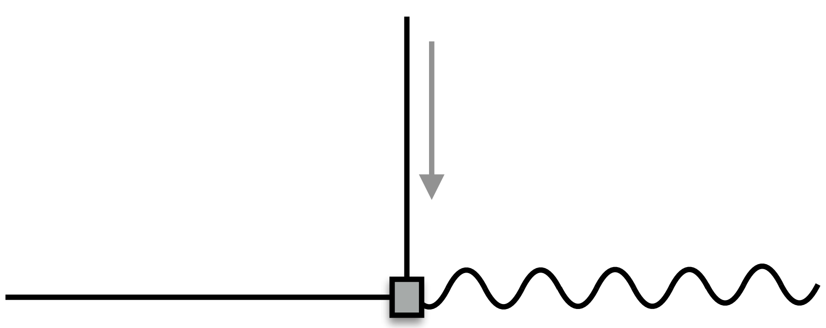
G is a smeared spectral function

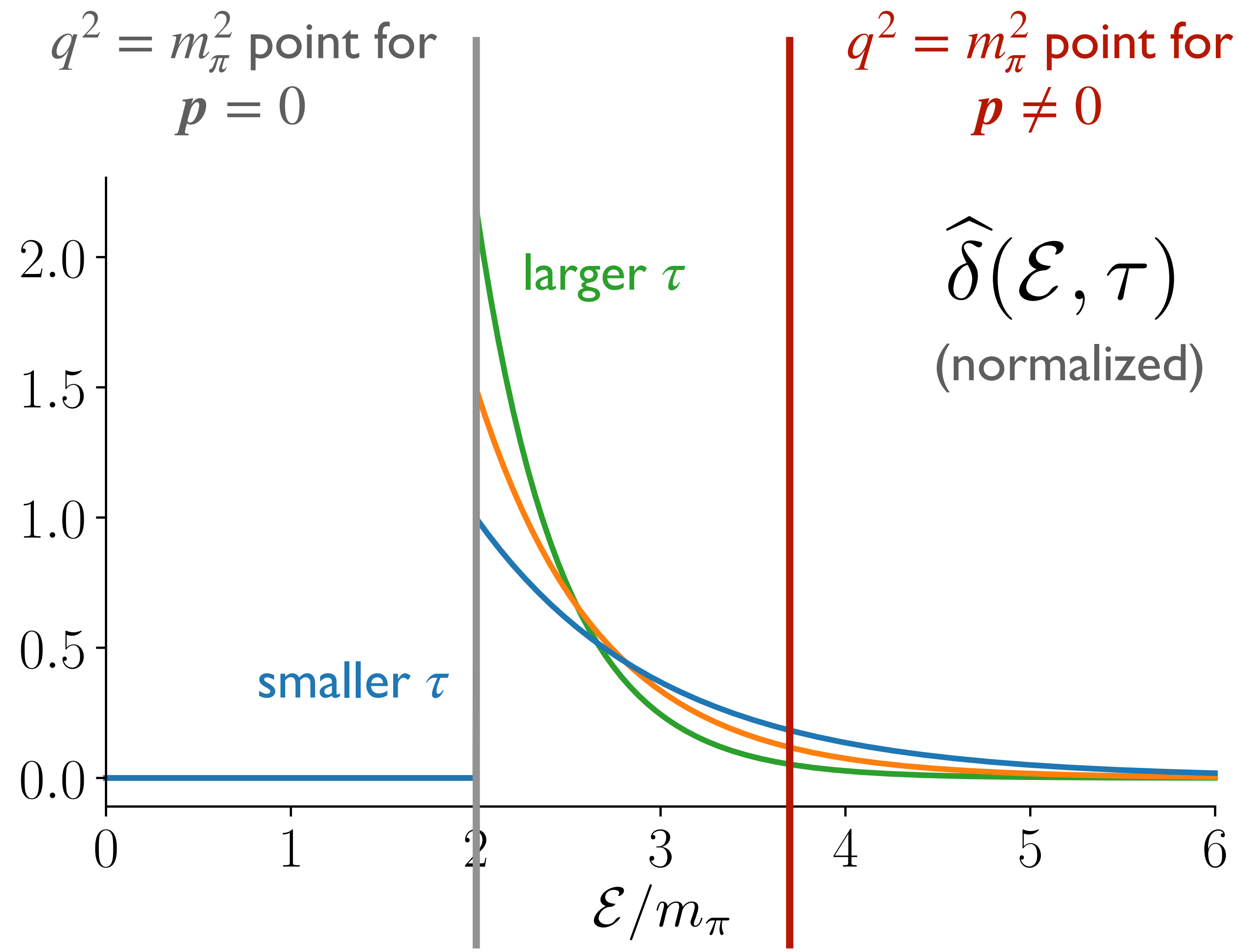
- Correlator can be viewed as a smeared spectral function

$$G_{[\boldsymbol{p}]}(\tau) = \int_0^\infty d\mathcal{E}\, \rho(\mathcal{E})\, \widehat{\delta}(\mathcal{E}, \tau) \qquad \widehat{\delta}(\mathcal{E}, \tau) = \theta(\mathcal{E} - 2m_\pi)\exp[-\mathcal{E}\tau]$$

$$\rho(\mathcal{E})$$

$$q^2 = (\mathcal{E} - \omega_{\boldsymbol{p}})^2 - \boldsymbol{p}^2$$

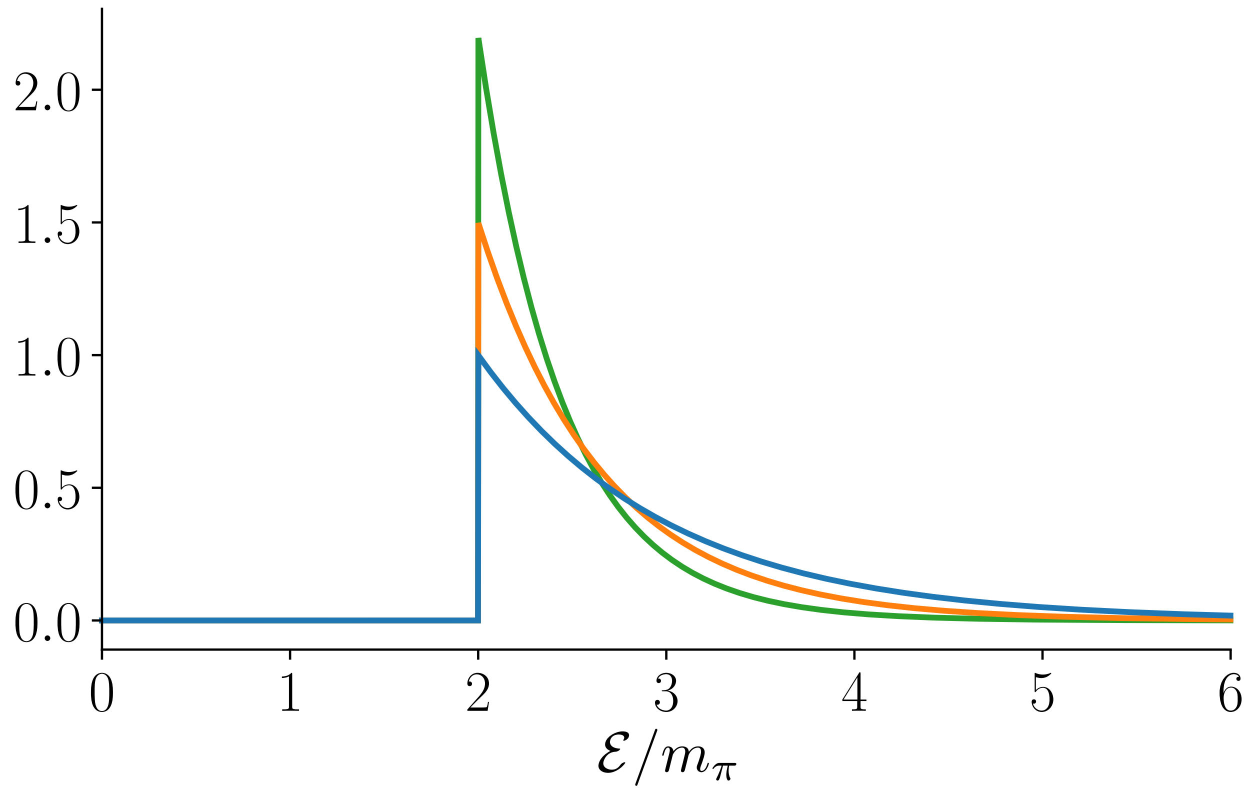




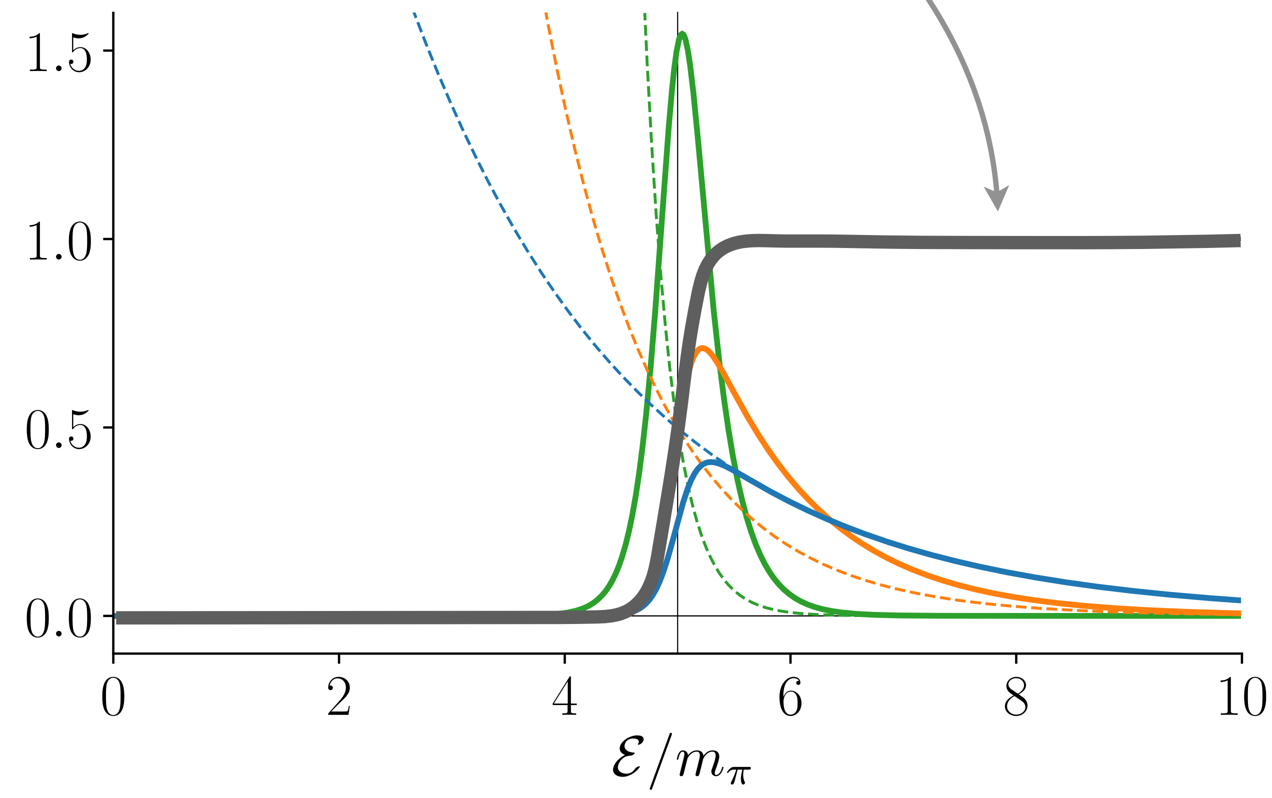
Bruno, MTH (2020)

Shift the peak!

$$\widehat{\delta}(\mathcal{E},\tau) = \theta(\mathcal{E} - 2m_\pi)\exp[-\mathcal{E}\tau]$$



$$\widehat{\delta}^{\Theta}(\mathcal{E},\tau) = \Theta(\mathcal{E} - 2m_\pi, \Delta)\exp[-\mathcal{E}\tau]$$



Bruno, MTH (2020)

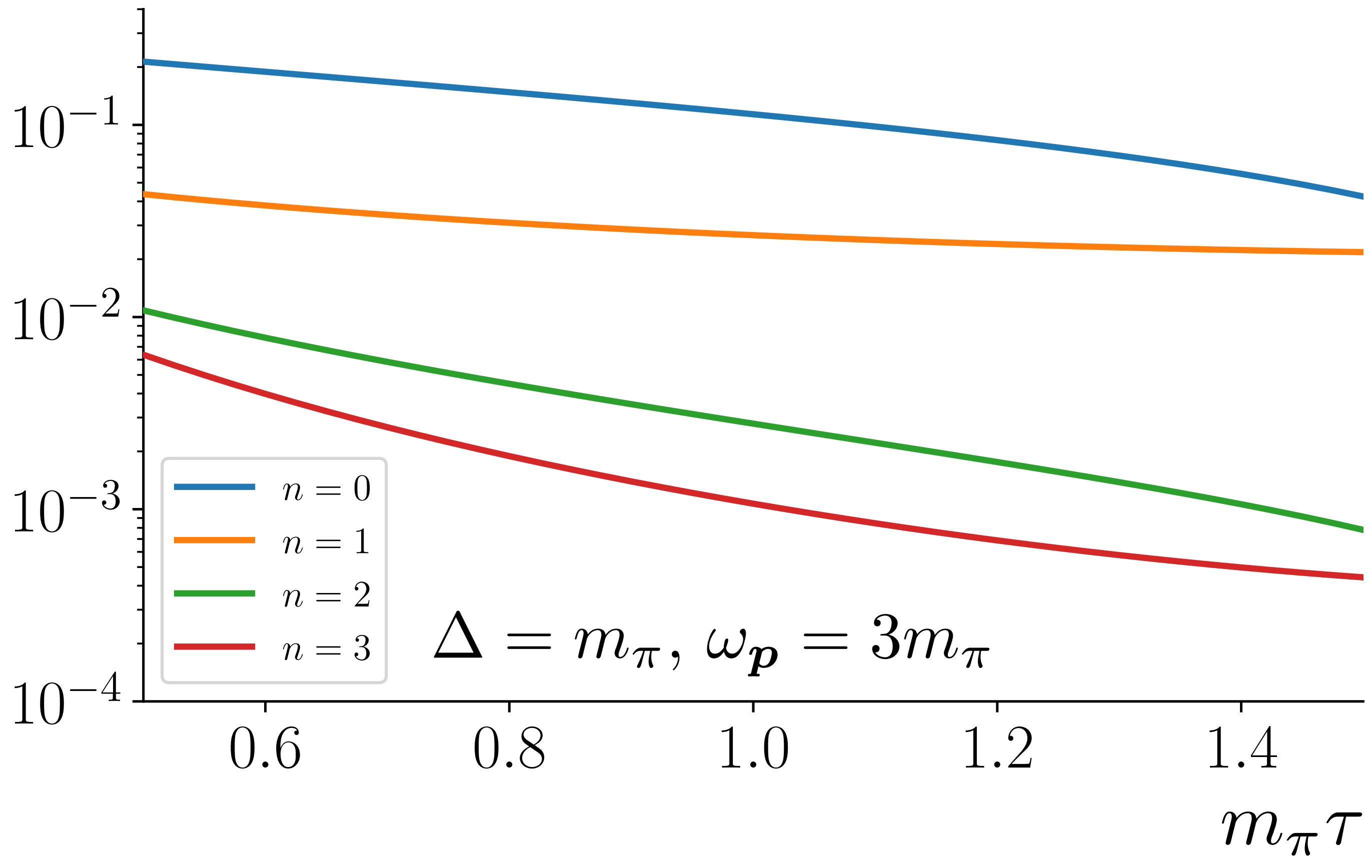
Reaching above threshold

- So the Maiani and Testa correlator becomes

$$G^{\Theta}_{[\boldsymbol{p}]}(\tau) = \langle \pi_{-\boldsymbol{p}} | \pi_{\boldsymbol{p}}(\tau)\, \Theta(\hat{H} - 2\omega_{\boldsymbol{p}}, \Delta)\, J(0) | 0 \rangle$$

- Now separating the fields gives something useful above threshold

$$G^{\Theta}_{[\boldsymbol{p}]}(\tau) = \frac{\sqrt{Z_\pi}}{2\omega_{\boldsymbol{p}}} e^{-\omega_{\boldsymbol{p}}\tau} \Big[\Theta(0,\Delta)\, \mathrm{Re} f(4\omega_{\boldsymbol{p}}^2) - 2\mathcal{J}^{(0)}(\tau,\Delta)\, \mathrm{Im} f(4\omega_{\boldsymbol{p}}^2) + \cdots \Big]$$



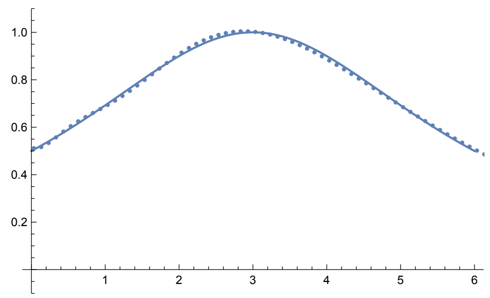
Hierarchy of $\mathscr{J}^{(n)}$ provides a useful fit function

known functional forms = distinction from the work with Bulava

Constructing the Θ correlator

$$G^{\Theta}_{[\boldsymbol{p}]}(\tau) = \langle \pi_{-\boldsymbol{p}} | \pi_{\boldsymbol{p}}(\tau)\, \Theta(\hat{H} - 2\omega_{\boldsymbol{p}}, \Delta)\, J(0) | 0 \rangle$$

- Two main methods:
  - Backus-Gilbert and HLT method
  - GEVP



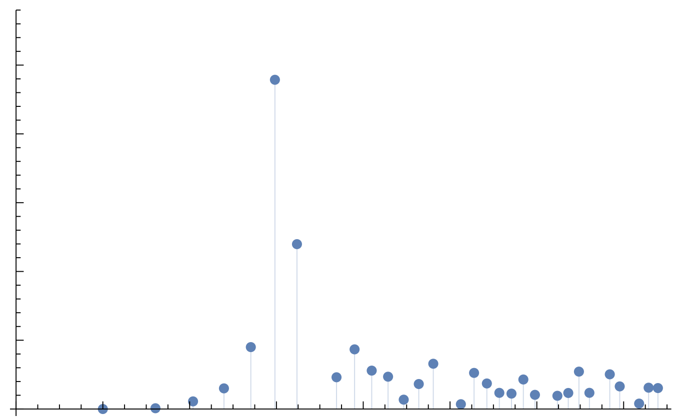
$$G^{\Theta}_{[\boldsymbol{p}]}(\tau) = \sum_n \Theta(E_n - 2\omega_{\boldsymbol{p}}, \Delta) e^{-(E_n - \omega_{\boldsymbol{p}})\tau} \langle \pi_{-\boldsymbol{p}} | \pi_{\boldsymbol{p}}(0) | n \rangle \langle n | J(0) | 0 \rangle$$

combine finite-volume energies and matrix elements

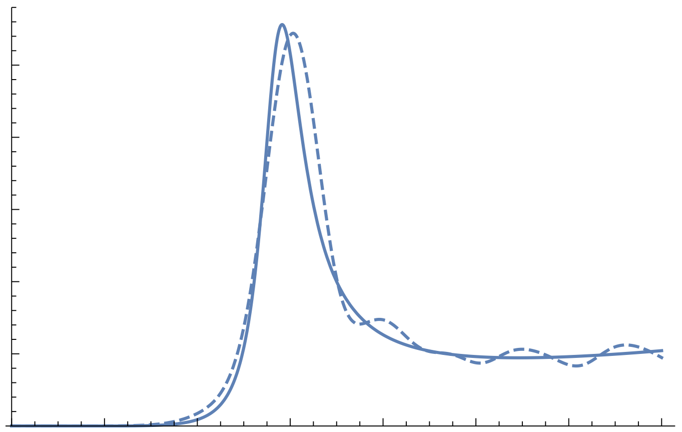
- Volume effects?… Suppressed as $e^{-\Delta L}$, but more investigation is needed

Conclusions

- Cannot solve the inverse problem, we can get $\widehat{\rho}_{L,\Delta}(\bar{\omega}) \equiv \int_0^\infty d\omega \, \widehat{\delta}_\Delta(\bar{\omega},\omega) \, \rho_L(\omega)$
- Smearing is needed anyway to ***suppress volume effects***



$$1/L \ll \Delta \ll \mu_{\text{physical}}$$



- Generalized Backus-Gilbert takes $\widehat{\delta}^{\text{target}}_\Delta(\bar{\omega},\omega)$ as input
- Recent work has connected this to the work of Maiani and Testa

This has unlocked a ***playground*** of calculations that we ***just beginning*** to explore

Thanks!