Multi-hadron observables from spectral functions

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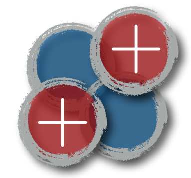
Recipe for strong force predictions

1. Lagrangian defining QCD
2. Formal / numerical machinery (lattice QCD)
3. A few experimental inputs (e.g. M_π, M_K, M_Ω)

$$\mathcal{L}_{\mathrm{QCD}} = \sum_f \overline{\Psi}_f (i\not{D} - m_f)\Psi_f - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a$$



Wide range of precision pre-/post-dictions

$m_q^{\overline{\mathrm{MS}}}(\mu)$ $\alpha_{\overline{\mathrm{MS}}}(\mu)$

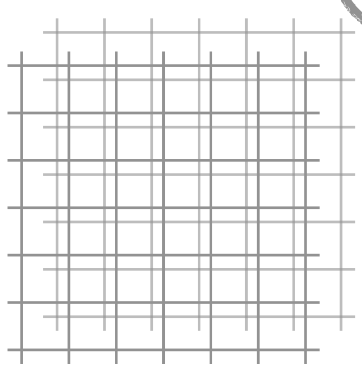
m_p

m_n f_π

g_A f_K

$(g-2)_\mu^{\mathrm{HVP}}$ f_D

f_B



FLAG 2019 Flavour Lattice Averaging Group

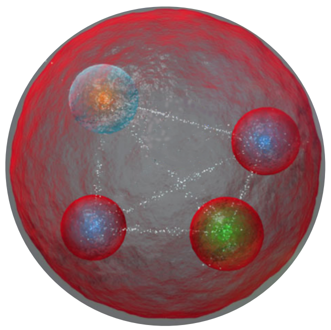
$\alpha^{(5)}_{\overline{\mathrm{MS}}}(M_Z) = 0.1182(8)$ lattice average

$\alpha^{(5)}_{\overline{\mathrm{MS}}}(M_Z) = 0.1174(16)$ PDG 18 (non-lattice)

Overwhelming evidence for QCD ✓ Tool for new-physics searches ✓

Multi-hadron observables

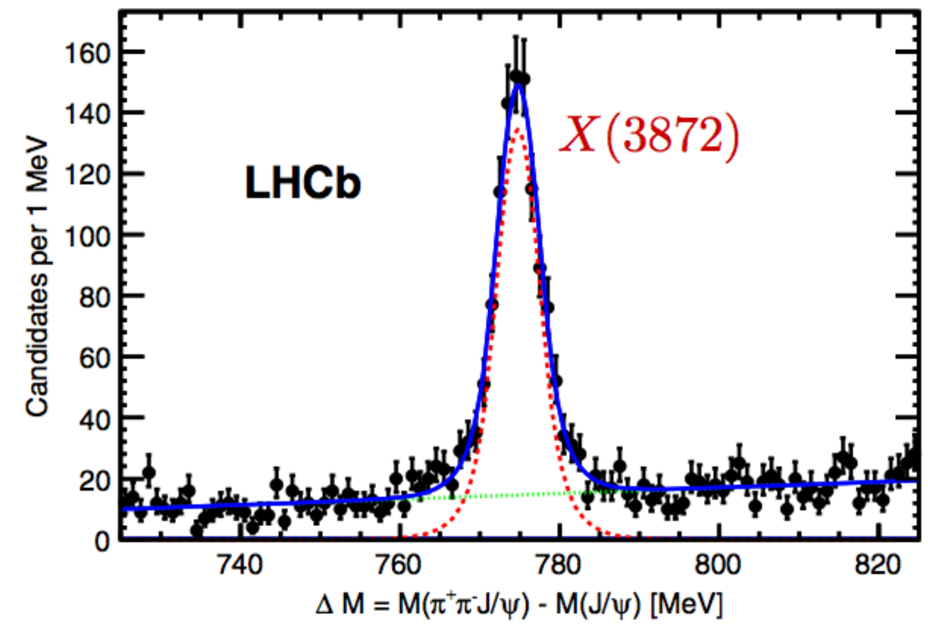
- Exotics, XYZs, tetra- and penta-quarks, *H* dibaryon



e.g. X(3872)

$\sim |D^0\overline{D}^{*0} + \overline{D}^0 D^{*0}\rangle$?

• LHCb (PRD92, 2015) •



- Eletroweak, CP violation, resonant enhancement

CP violation in charm $D \to \pi\pi, K\overline{K}$

$$\Delta A_{CP} = (-15.4 \pm 2.9) \times 10^{-4}$$

• LHCb (PRL, 2019) •

$f_0(1710)$ could enhance ΔA_{CP}

• Soni (2017) •

Resonant B decays $B \to K^* \ell\ell \to K\pi\,\ell\ell$

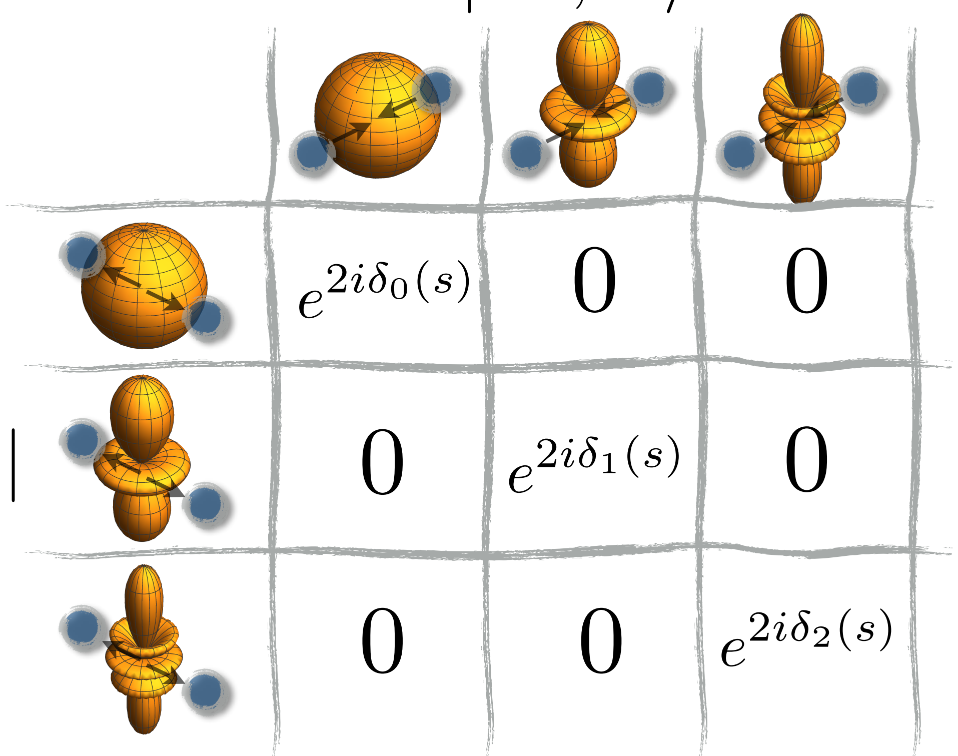
$|X\rangle, |\rho\rangle, |K^*\rangle, |f_0\rangle \quad \notin$ **QCD Fock space**

QCD Fock space

- At low-energies QCD = hadronic degrees of freedom $\pi \sim \bar{u}d,\ K \sim \bar{s}u,\ p \sim uud$
- Overlaps of multi-hadron ***asymptotic states*** → S matrix

$|\pi\pi, \text{in}\rangle$

$S(s) \equiv \langle \pi\pi, \text{out}|$



depends on $s = E_{\text{cm}}^2$
and angular variables

diagonal in angular momentum

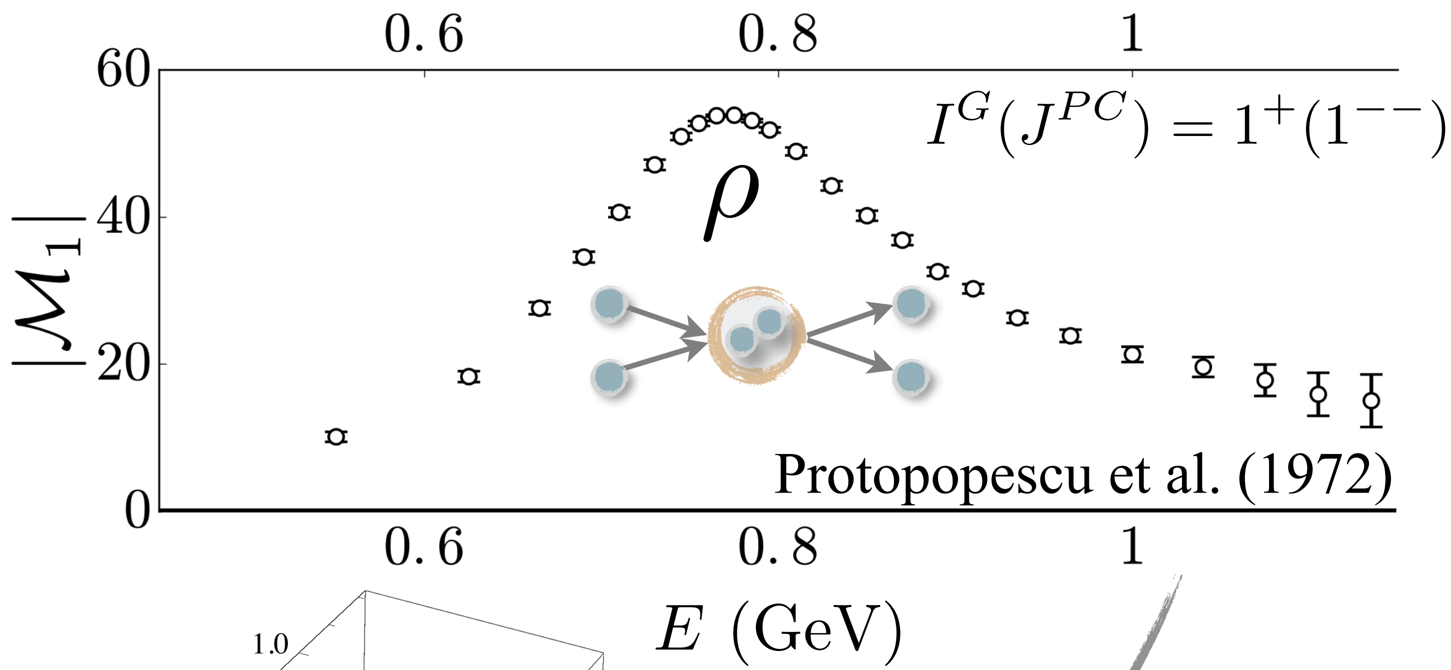
$$\mathcal{M}_\ell(s) \propto e^{2i\delta_\ell(s)} - 1$$

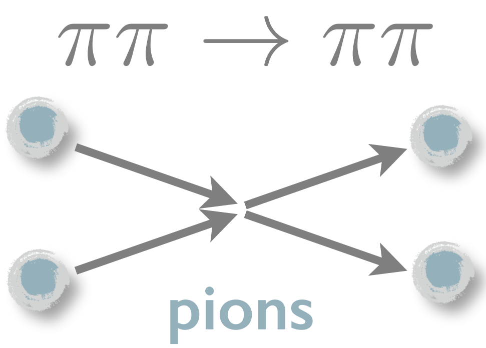
- ***An enormous space of information*** $|\pi\pi\pi\pi, \text{in}\rangle \quad |K\overline{K}, \text{in}\rangle \quad \cdots$

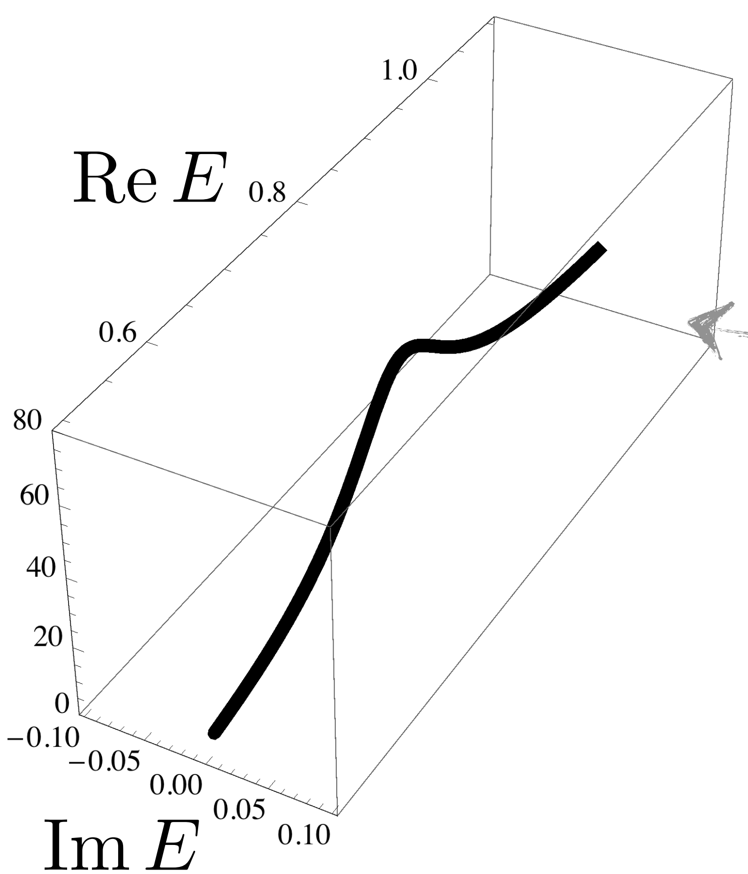
QCD resonances

- Roughly speaking, a bump in: $|\mathcal{M}_\ell(s)|^2 \propto |e^{2i\delta_\ell(s)} - 1|^2 \propto \sin^2 \delta_\ell(s)$

scattering rate







Extend to the
complex plane

QCD resonances

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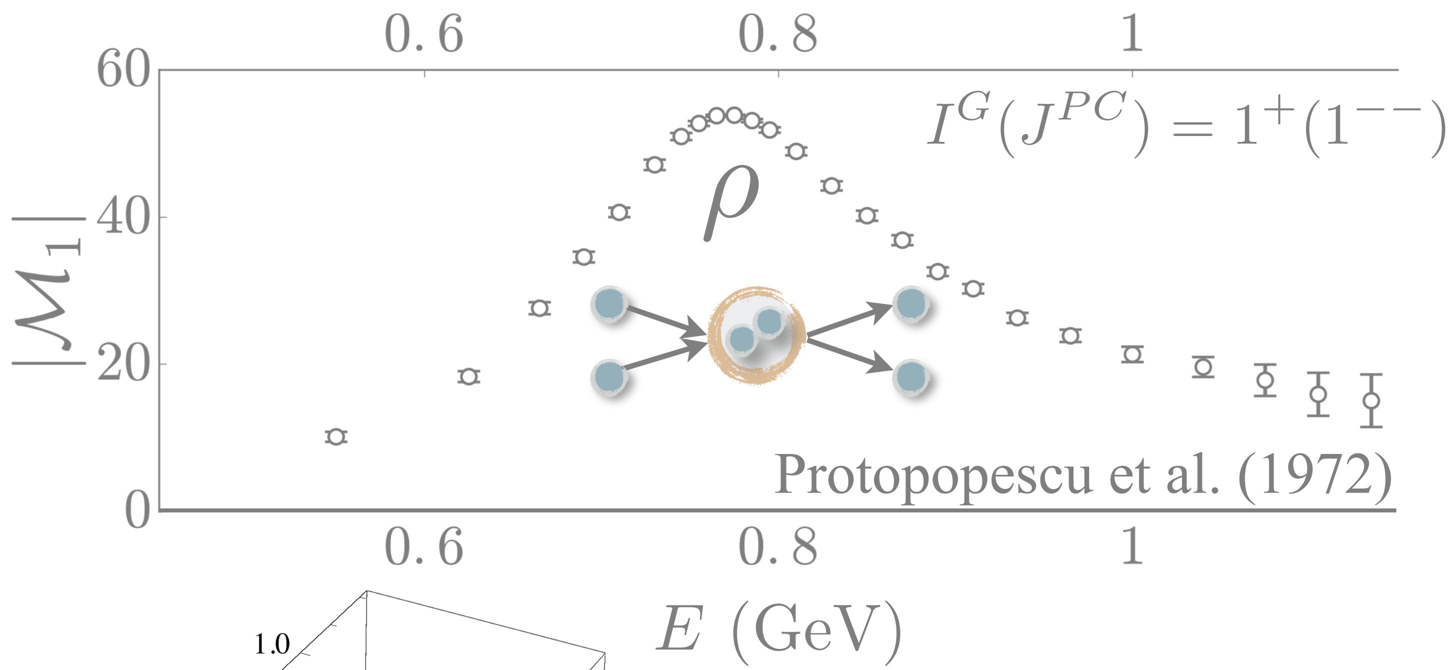
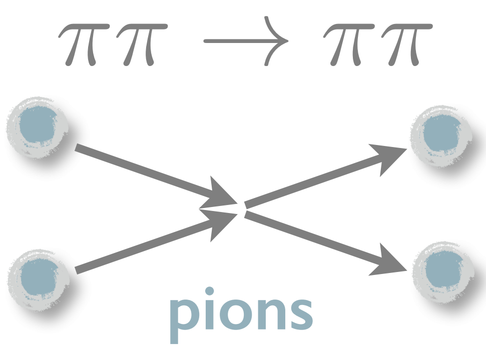
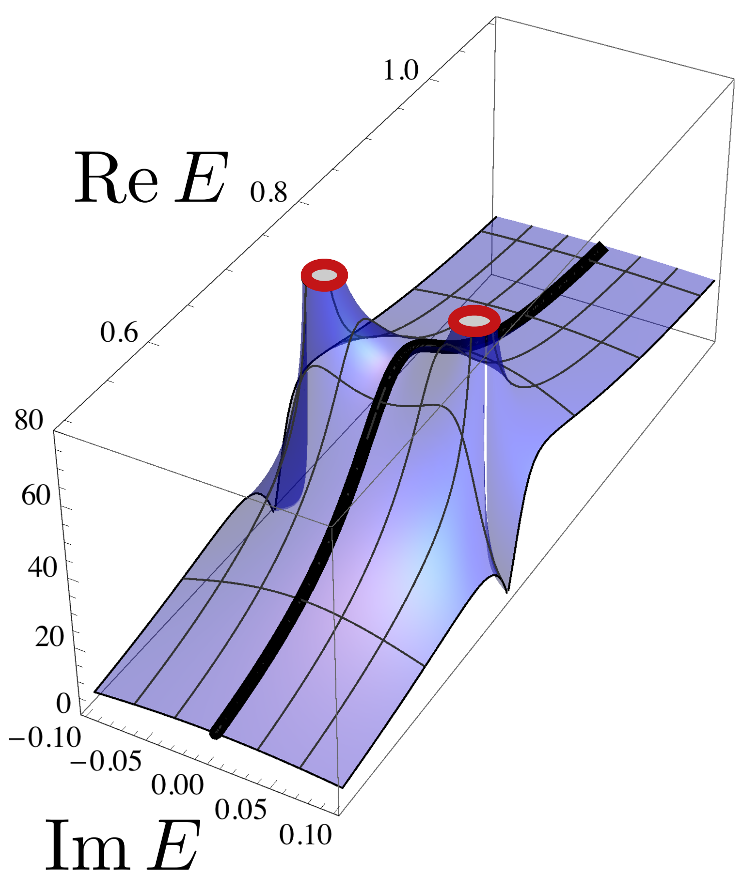
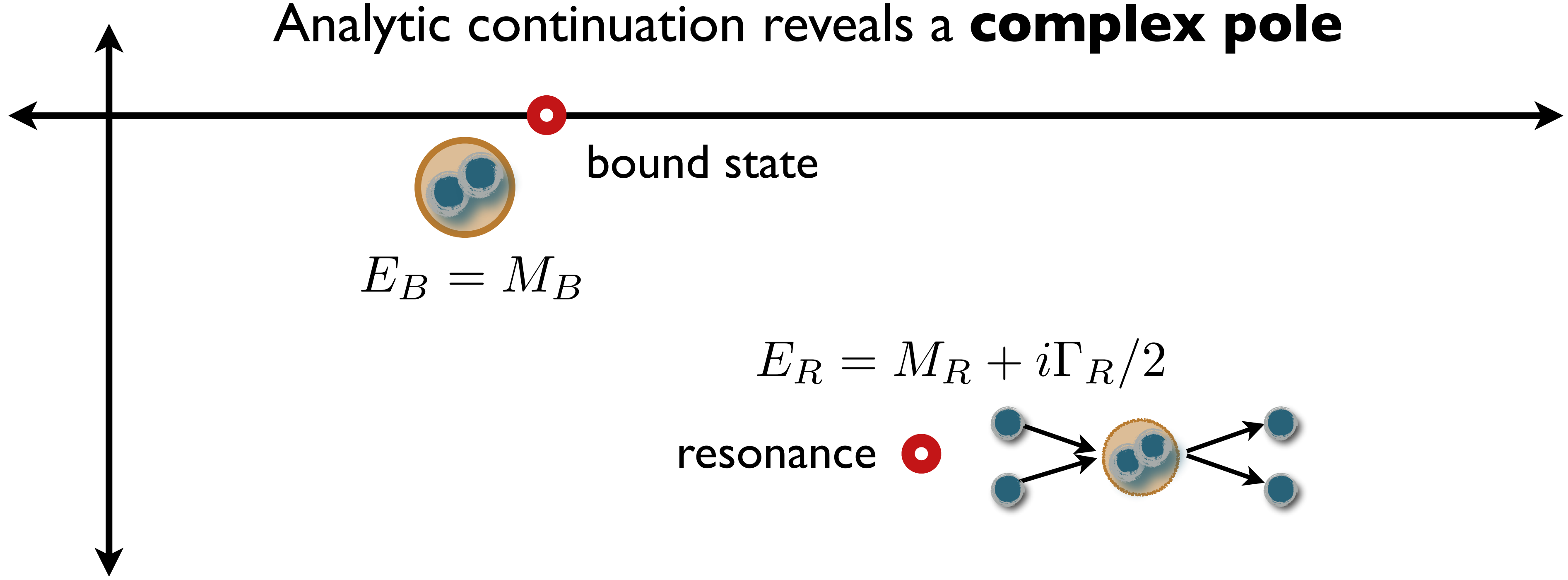
scattering rate





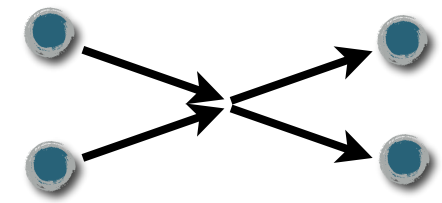


Analytic continuation reveals a **complex pole**



Analyticity

- Instead of $|\mathcal{M}(s)|^2$ → analytically continue the ***amplitude*** itself



  For two-particle energies $(2m)^2 < s < (4m)^2$, what is the analytic structure?

- The optical theorem tells us…

$$\rho(s)|\mathcal{M}_\ell(s)|^2 = \mathrm{Im}\,\mathcal{M}_\ell(s)$$

where $\rho(s) = \dfrac{\sqrt{1-4m^2/s}}{32\pi}$ is the two-particle phase space

- Unique solution is…

$$\mathcal{M}_\ell(s) = \frac{1}{\mathcal{K}_\ell(s)^{-1} - i\rho(s)}$$

K matrix (short distance)

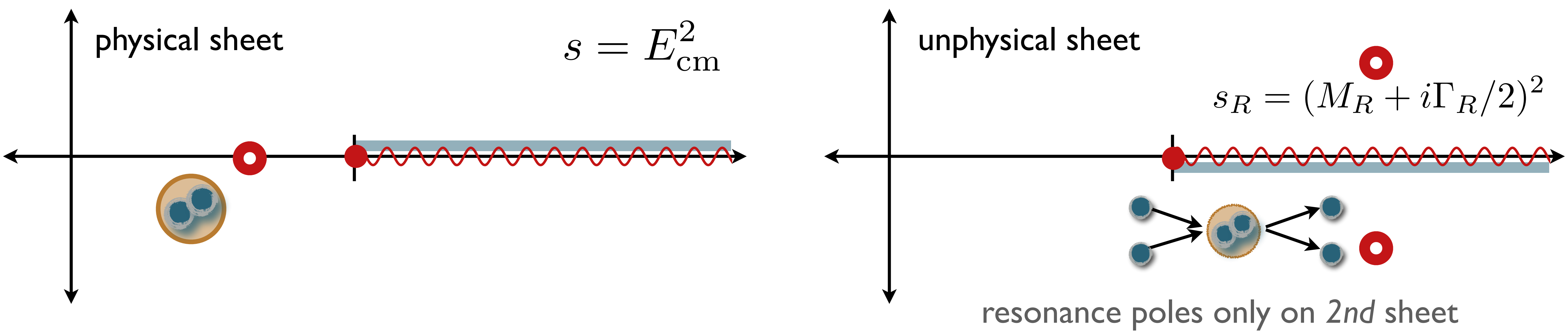
phase-space cut (long distance)

Key message: *The scattering amplitude has a square-root branch cut*

Cuts and sheets

$$\mathcal{M}_\ell(s) = \frac{1}{\mathcal{K}_\ell(s)^{-1} - \rho(s)} \propto \frac{1}{p\cot\delta_\ell(s) - ip} \propto e^{2i\delta_\ell(s)} - 1 \qquad \rho(s) \propto i\sqrt{s-(2m)^2}$$

- ❑ Each channel generates a *square-root cut* → doubles the number of sheets



- ❑ Important lessons:

Details of analyticity = important for quantitative understanding

Possible to separate…

(i) long-distance kinematic singularities

(ii) short-distance/microscopic physics (depending on interaction details)

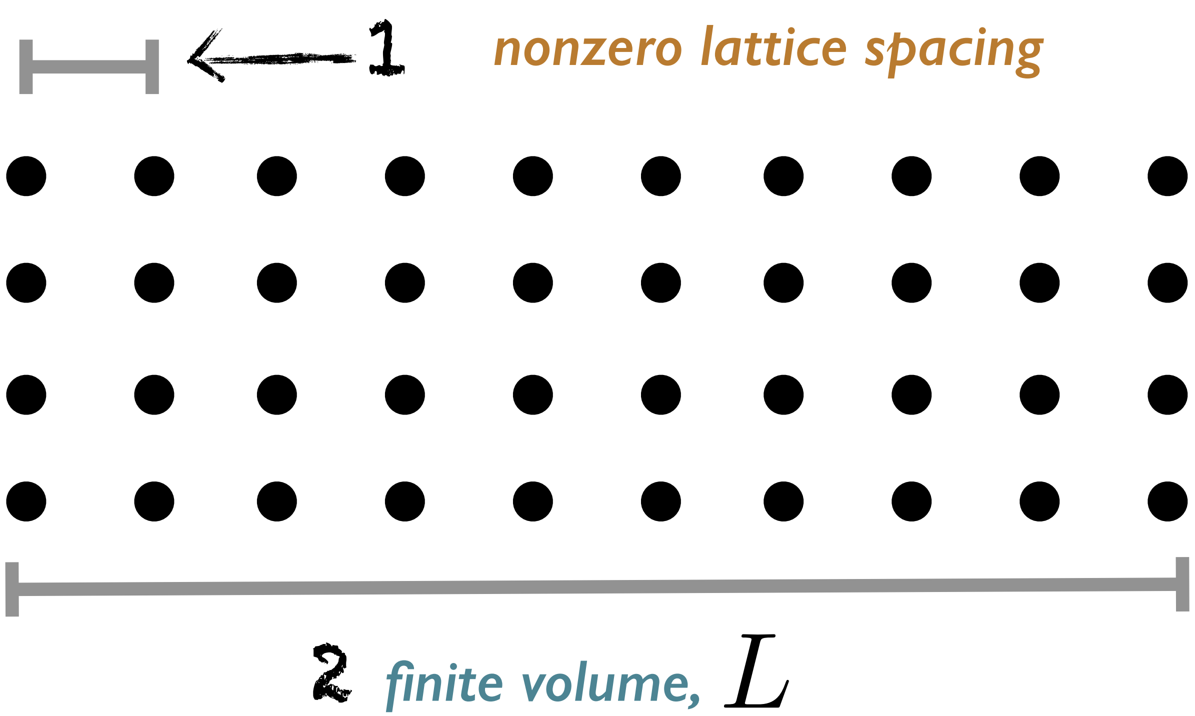
Microscopic physics *via Lattice QCD*

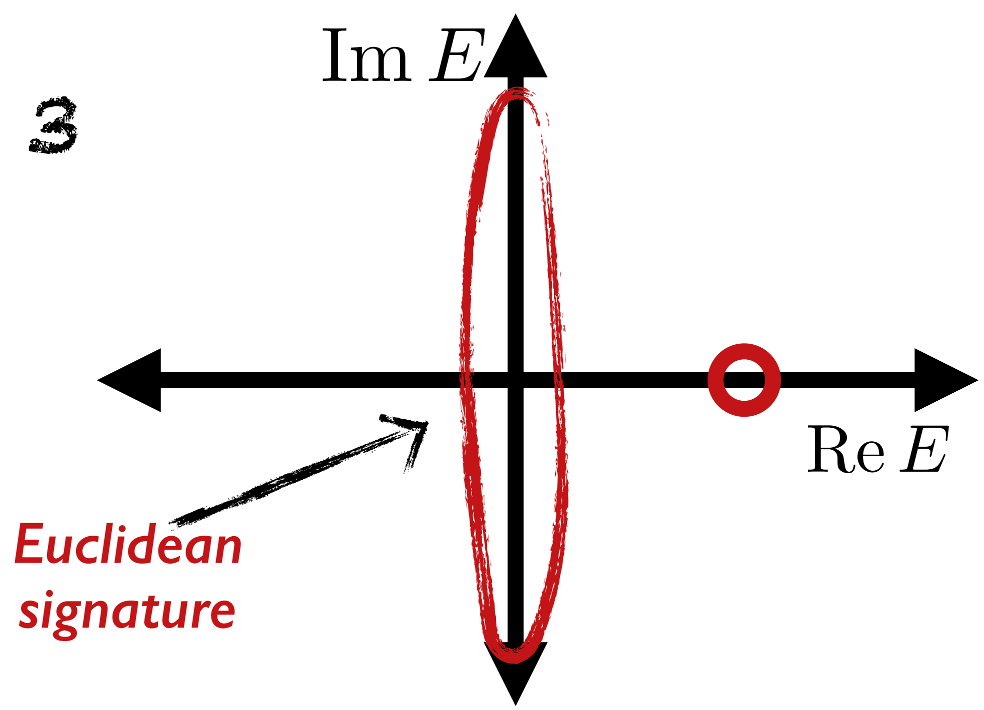
$$\text{observable} = \int \mathcal{D}\phi \; e^{iS} \left[\begin{array}{c} \textbf{interpolator} \\ \textbf{for observable} \end{array} \right]$$

Microscopic physics *via Lattice QCD*

$$\text{observable?} = \int d^N\phi \, e^{-S} \left[\begin{array}{c} \textbf{interpolator} \\ \textbf{for observable} \end{array} \right]$$

To proceed we have to make *three modifications*



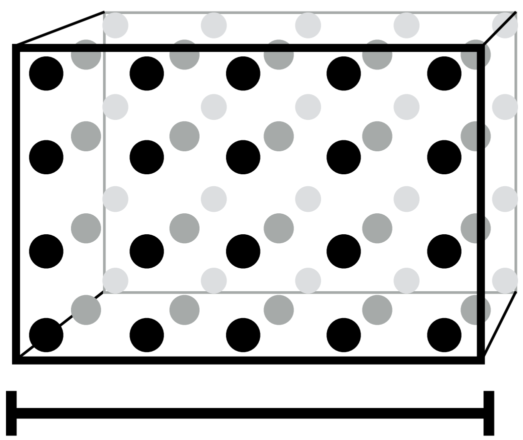


Also… $M_{\pi,\text{lattice}} > M_{\pi,\text{our universe}}$

(but physical masses → increasingly common)

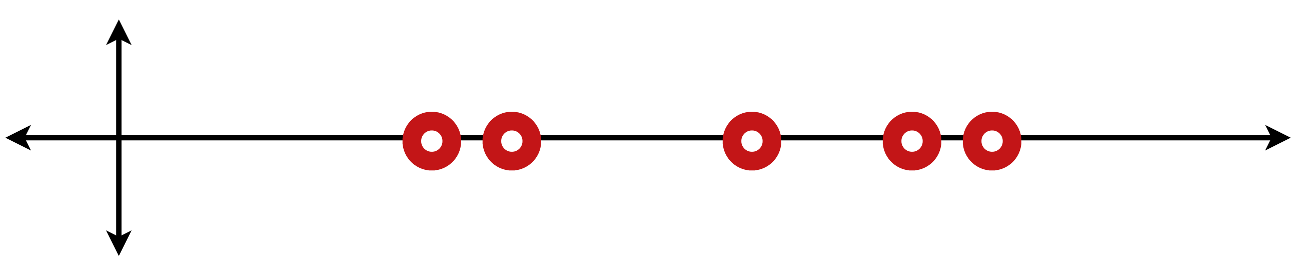
Difficulties for multi-hadron observables

- The ***Euclidean signature***…
  - *Obscures* real time evolution (that defines scattering)
  - *Prevents* normal LSZ (want $p_4^2 = -(\mathbf{p}^2 + m^2)$, but we have only $p_4^2 > 0$)

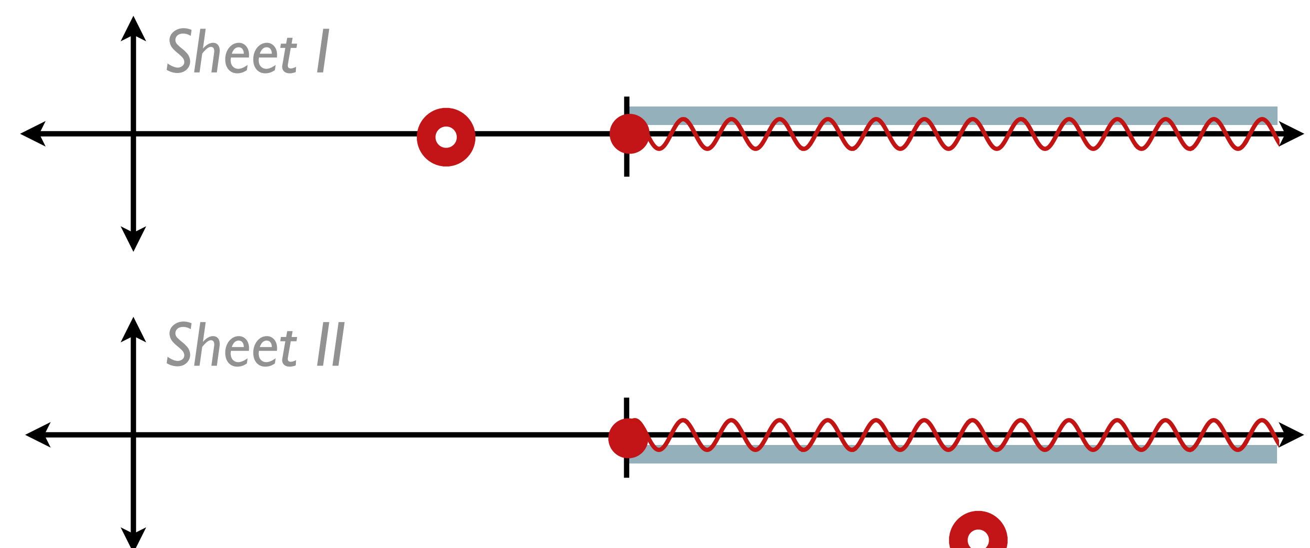


- The ***finite volume***…
  - *Discretizes* the spectrum
  - *Eliminates* the branch cuts and extra sheets
  - *Hides* the resonance poles

Finite-volume analytic structure



Infinite-volume analytic structure



Two strategies…

- Finite-volume as a tool
  - LQCD → Energies and matrix elements

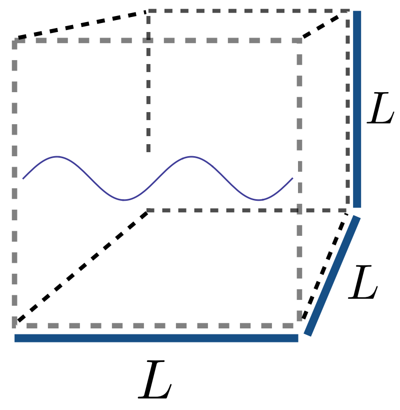
$$\langle \mathcal{O}_j(\tau)\mathcal{O}_i^\dagger(0)\rangle = \sum_n \langle 0|\mathcal{O}_j(\tau)|E_n\rangle\langle E_n|\mathcal{O}_i^\dagger(0)|0\rangle = \sum_n e^{-E_n(L)\tau} Z_{n,j} Z_{n,i}^*$$

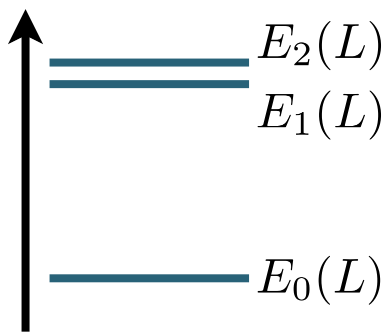
  - Our task is relate $E_n(L)$ and $\langle E_{m'}|\mathcal{J}(0)|E_m\rangle$ to ***experimental observables***
  - Applicable only in limited energy range for two- and three-hadron states

- Spectral function method
  - Formally applies for any number of particles / any energy range
  - An answer to the question… “Can’t you just analytically continue?”
  - Still important challenges and limitations to consider

The finite-volume as a tool

- Finite-volume set-up



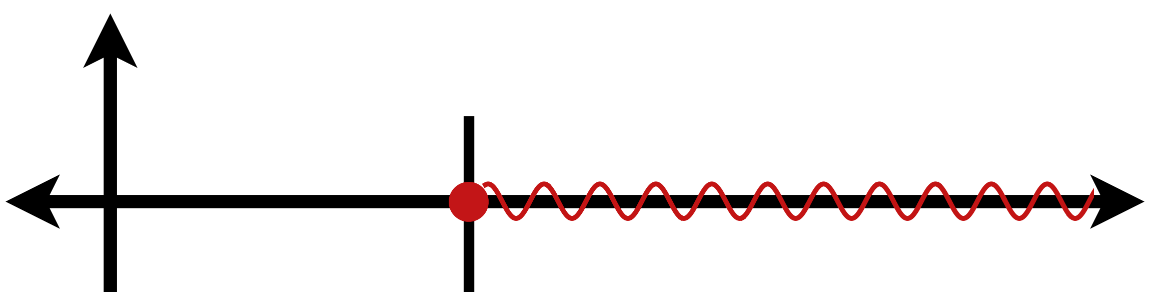


- cubic, spatial volume (extent L)
- periodic

$$\vec{p} = \frac{2\pi}{L}\vec{n}\,, \qquad \vec{n} \in \mathbb{Z}^3$$

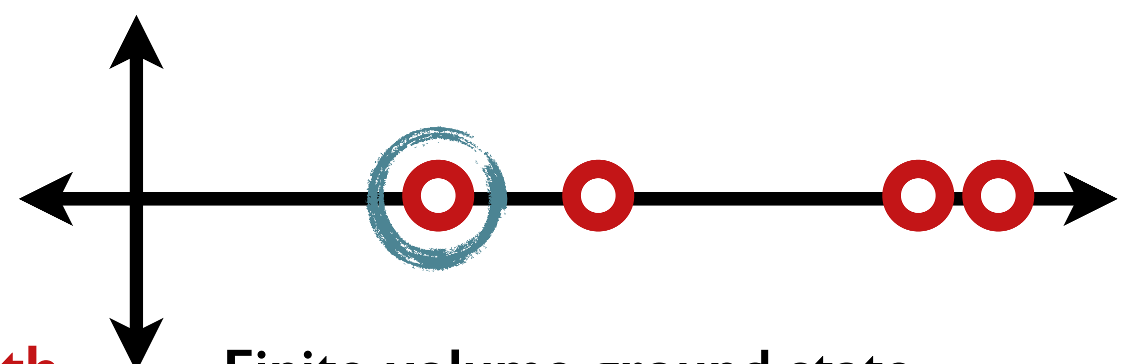
- L is large enough to neglect $e^{-M_\pi L}$
- T and lattice also negligible

- Scattering leaves an *imprint* on finite-volume quantities



$E_0 = 2M_\pi$

Infinite-volume threshold



Finite-volume ground state

scattering length

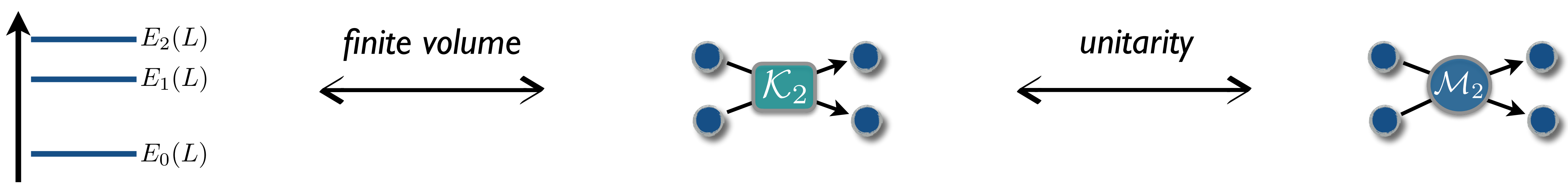
$$\mathcal{M}_{\ell=0}(2M_\pi) = -32\pi M_\pi a$$

$$E_0(L) = 2M_\pi + \frac{4\pi a}{M_\pi L^3} + \mathcal{O}(1/L^4)$$

• Huang, Yang (1958) •

General method

$$\det[\mathcal{K}^{-1}(s) + F(P,L)] = 0 \qquad F(P,L) \equiv \text{Matrix of known geometric functions}$$



Holds only for two-particle energies $s < (4m)^2$ Neglects e^{-mL}

Generalized to *non-degenerate masses, multiple channels, spinning particles*

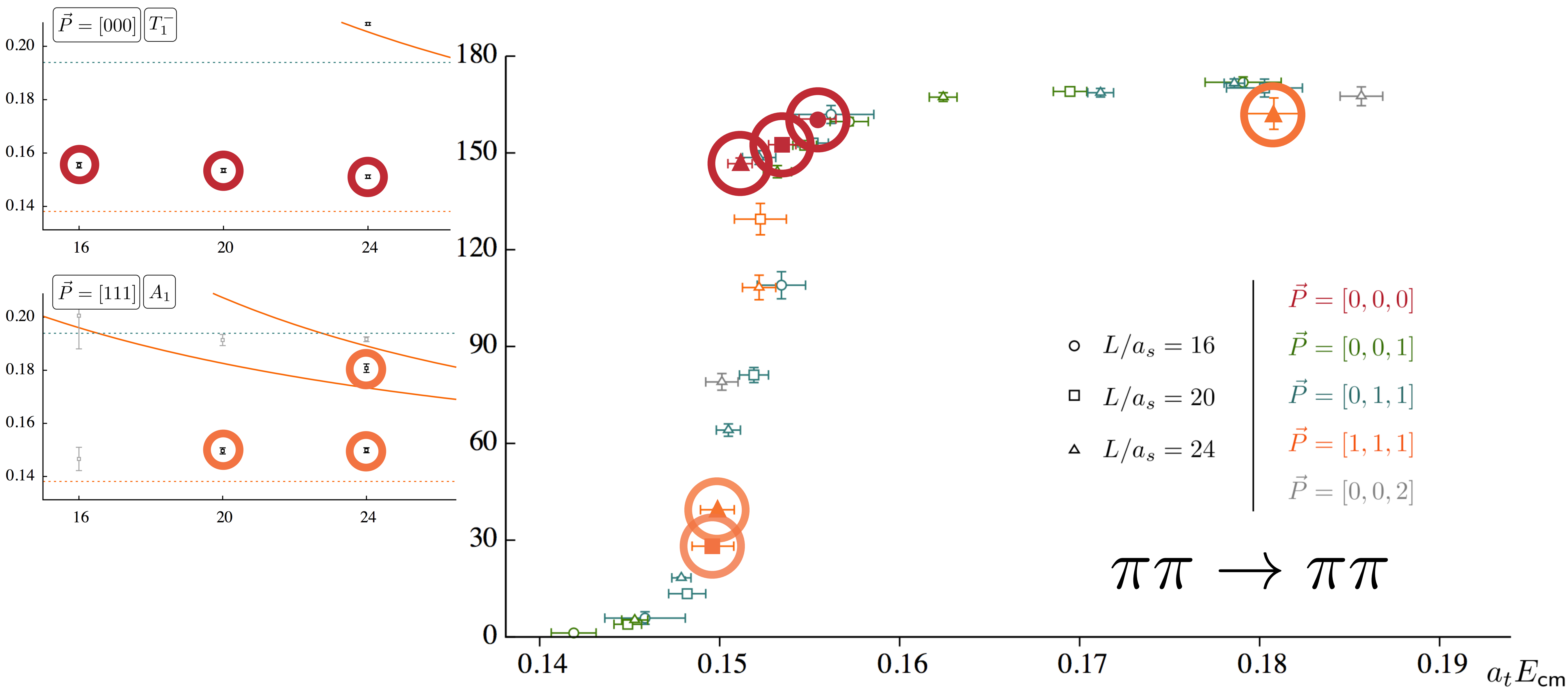
Encodes angular momentum mixing

Huang, Yang (1958) • Lüscher (1986, 1991) • Rummukainen, Gottlieb (1995)
Kim, Sachrajda, Sharpe (2005) • Christ, Kim, Yamazaki (2005) • He, Feng, Liu (2005)
Leskovec, Prelovsek (2012) • Bernard *et. al.* (2012) • MTH, Sharpe (2012) • Briceño, Davoudi (2012)
Li, Liu (2013) • Briceño (2014)

Using the result

- Single-channel case *(pions in a p-wave)*

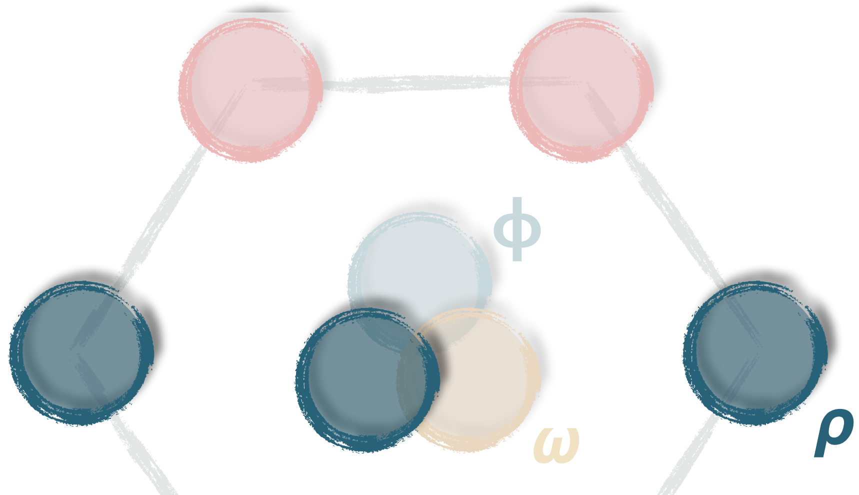
$$\mathcal{K}(s_n)^{-1} = \rho \cot\delta(s_n) = -F(E_n, \vec{P}, L)$$

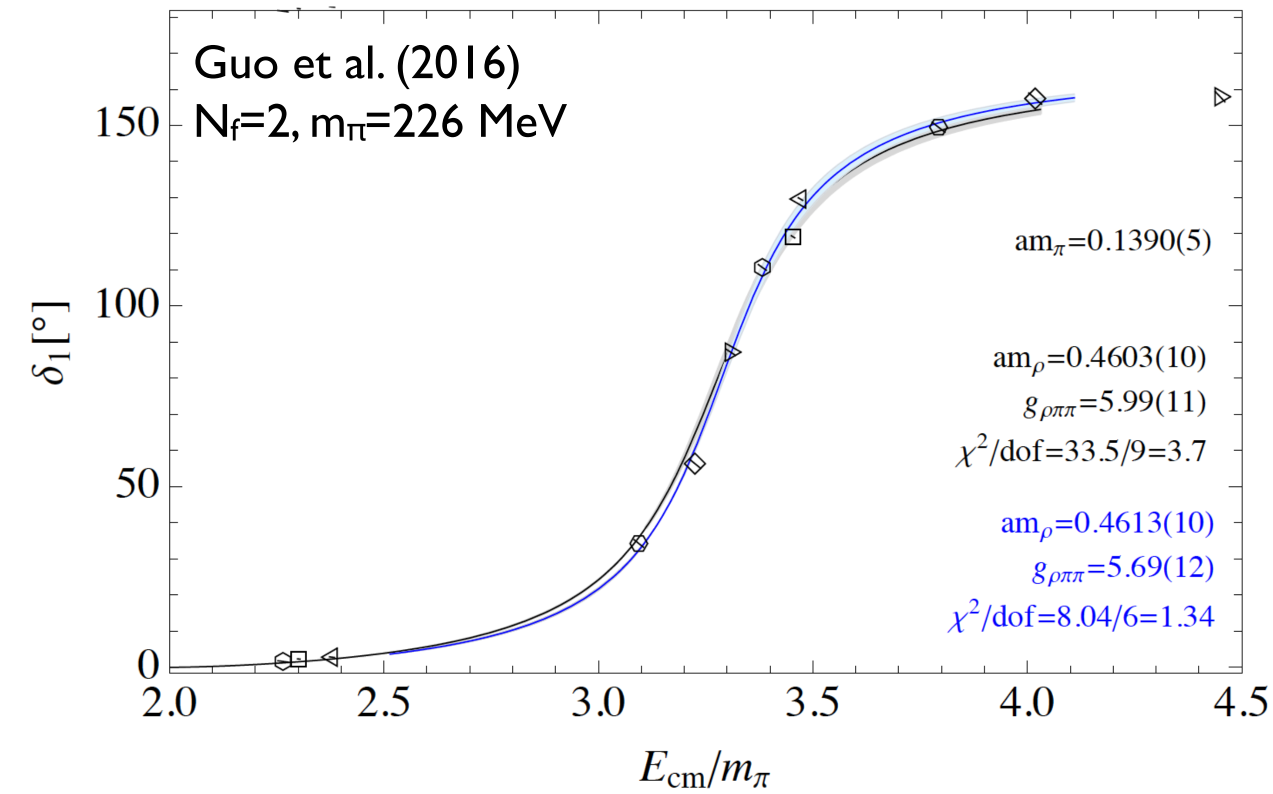


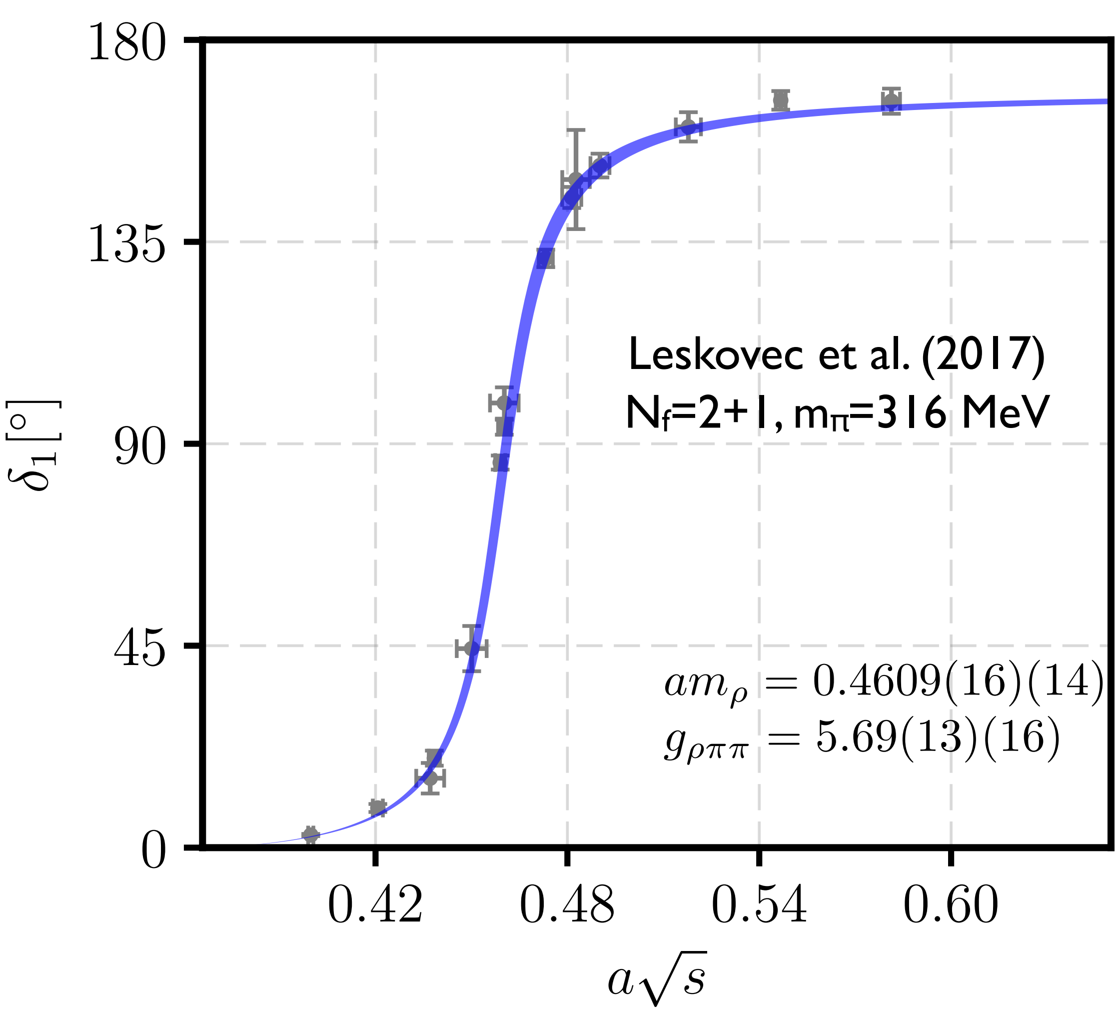
• Dudek, Edwards, Thomas in *Phys.Rev.* D87 (2013) 034505 •

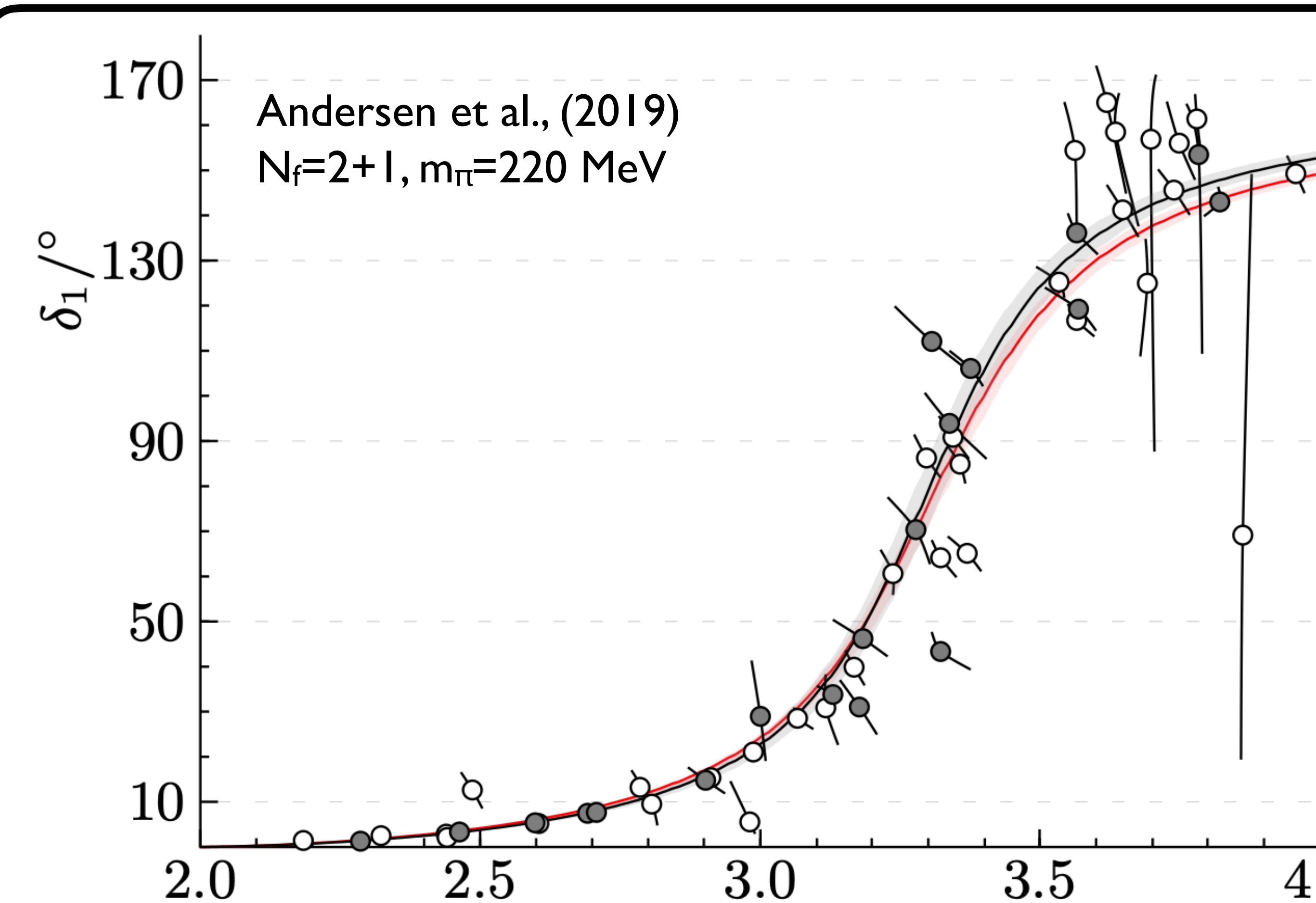
$$\rho \to \pi\pi$$

$$I^G(J^{PC}) = 1^+(1^{--})$$







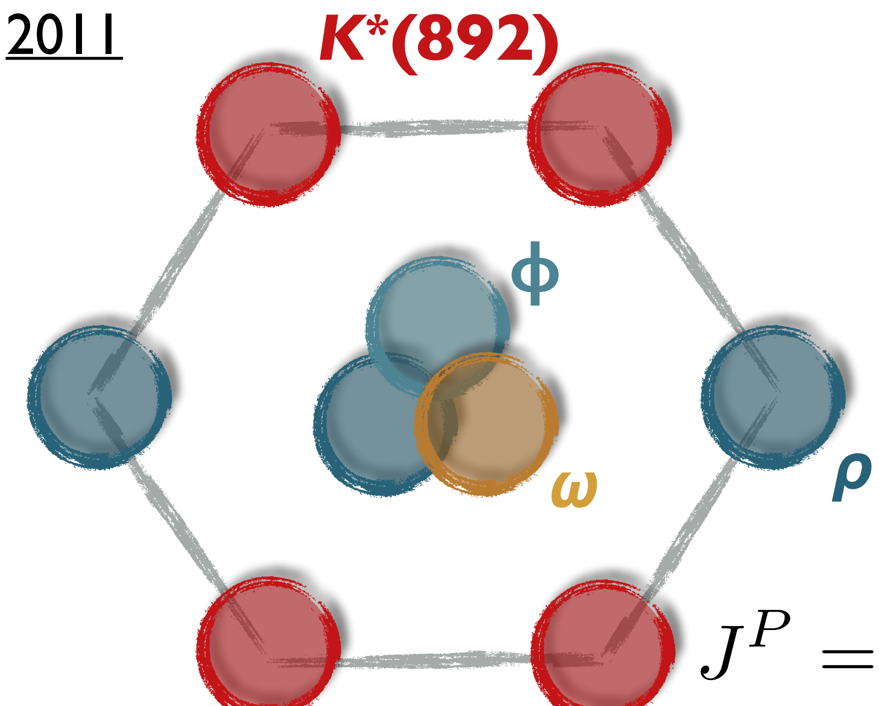


$\rho \to \pi\pi$

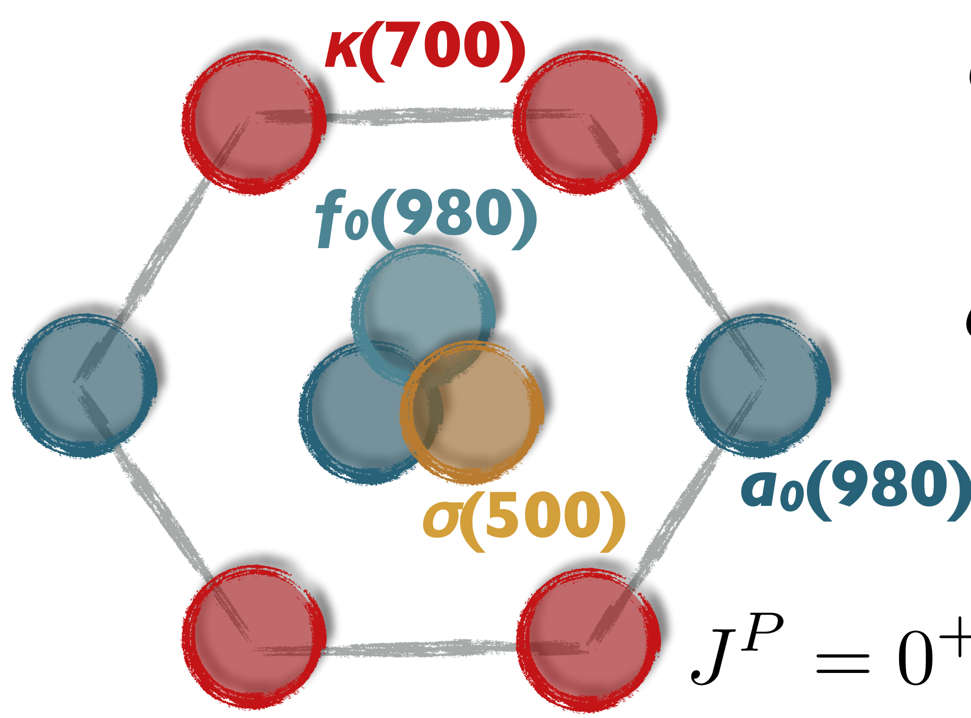
- CP-PACS/PACS-CS 2007, 2011
- ETMC 2010
- Lang et al. 2011
- HadSpec 2012, **2016**
- Pellisier 2012
- RQCD 2015
- Guo et al. 2016
- Fu et al. 2016
- Bulava et al. 2016
- Alexandrou et al. 2017
- Andersen et al. 2018
- Fischer et al. 2020
- Erben et al. 2020

$\sigma \to \pi\pi$

- Prelovsek et al. 2010
- Fu 2013
- Wakayama 2015
- Howarth and Giedt 2017
- Briceño et al. 2017
- Guo et al. 2018



$J^P = 1^-$



$J^P = 0^+$

$\kappa \to K\pi$

$K^* \to K\pi$

- Lang et al. 2012
- Prelovsek et al. 2013
- **Wilson et al. 2015**
- RQCD 2015
- **Brett et al. 2018**
- Wilson et al. 2019
- Rendon et al. 2020

$b_1 \to \pi\omega, \pi\phi$

- **Woss et al. 2019**

$a_0(980) \to \pi\eta, K\overline{K}$

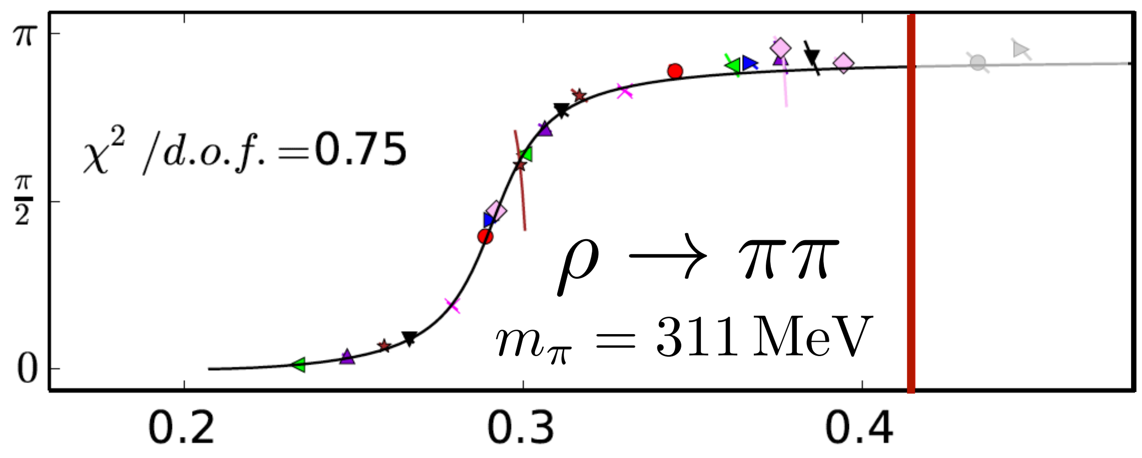
- **Dudek et al. 2016**

$\sigma, f_0, f_2 \to \pi\pi, K\overline{K}, \eta\eta$

- **Briceño et al. 2017**

See the recent review by Briceño, Dudek and Young

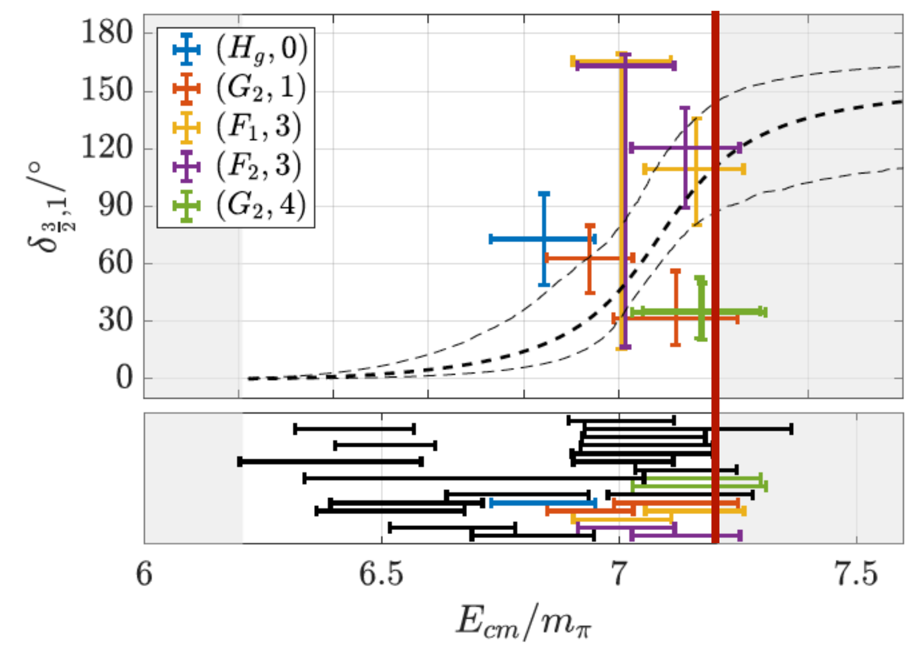
Multi-particle barrier



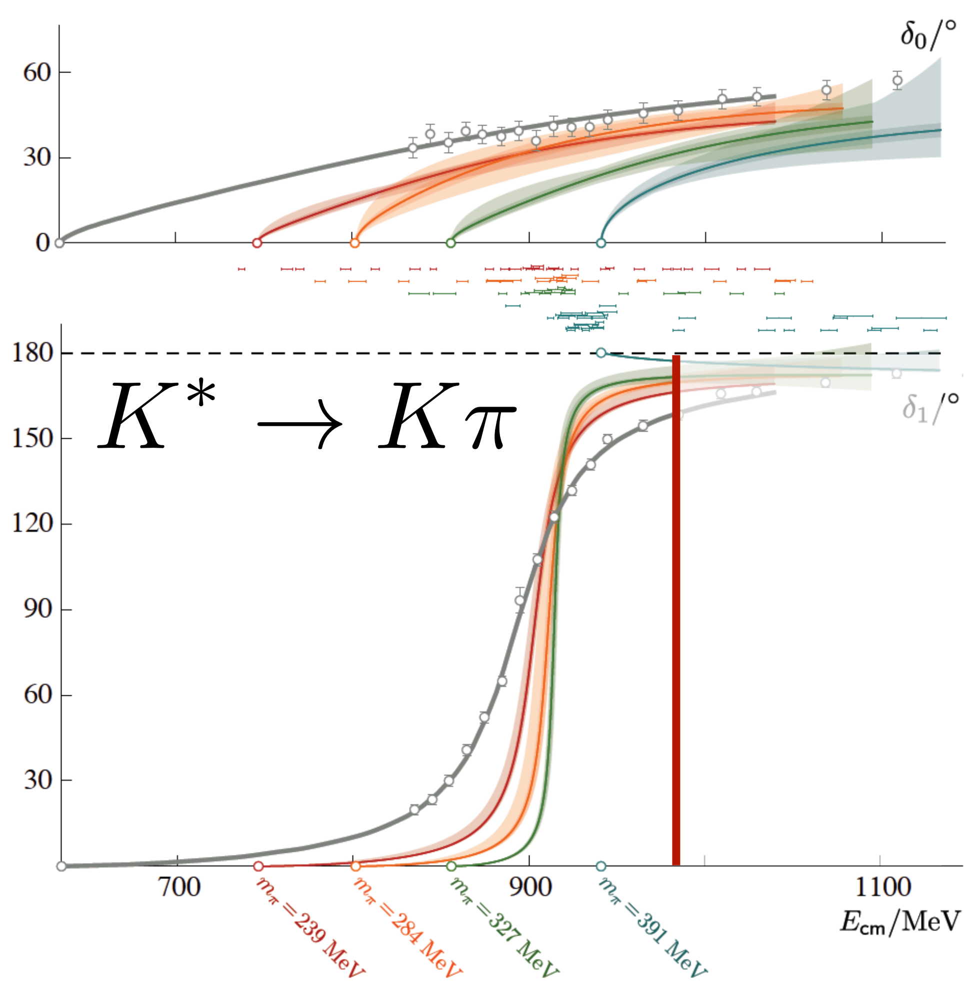
• Erben et al. (2019) •

$\Delta \to N\pi$

$m_\pi = 200\,\mathrm{MeV}$



• Andersen et al. (2019) •



• Wilson et al. *PRL123* (2019) •

Formalism only strictly holds up to lowest 3- or 4-particle threshold

Major progress in three-particle states

MTH, Sharpe, Briceño, Mai, Döring, Rusetsky…

… but is there another way?

Two strategies…

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$$\langle \mathcal{O}_j(\tau)\mathcal{O}_i^\dagger(0)\rangle = \sum_n \langle 0|\mathcal{O}_j(\tau)|E_n\rangle\langle E_n|\mathcal{O}_i^\dagger(0)|0\rangle = \sum_n e^{-E_n(L)\tau} Z_{n,j} Z_{n,i}^*$$

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Correlation functions → observables

- Lattice QCD gives finite-volume Euclidean correlators

$$\langle 0|\, \mathcal{O}_1(0)\, e^{-\hat{H}\tau}\, \mathcal{O}_2(0)\, |0\rangle_L$$

have

- Complete physical information is contained in…

$$\langle 0|\, \mathcal{O}_1(0)\, f(\hat{H})\, \mathcal{O}_2(0)\, |0\rangle_\infty$$

want

- Detailed choice of *f(E)* and operators determines the observable

R-ratio

$$\langle 0|\, j_\mu(0)\, \delta(\hat{H} - \omega)\, j_\mu(0)\, |0\rangle_\infty$$

Meyer • Bailas, Hashimoto, Ishikawa (2020) • Alexandrou et al. (2022) •

D-meson total lifetime

$$\langle D|\, \mathcal{H}_W(0)\, \delta(M_D - \hat{H})\, \mathcal{H}_W(0)\, |D\rangle_\infty$$

• MTH, Meyer, Robaina (2017) • K.F.-Liu (2016) • Hashimoto (2019-2021)

ππ → ππ amplitude

$$\langle \pi|\, \pi(0)\, \frac{1}{E - \hat{H} + i\epsilon}\, \pi(0)\, |\pi\rangle_\infty$$

j → ππ amplitude

$$\langle \pi|\, \pi(0)\, \frac{1}{E - \hat{H} + i\epsilon}\, j_\mu(0)\, |0\rangle_\infty$$

Bulava, MTH (2019)

Linear reconstruction (1/2)

$$\underset{\text{have}}{\langle \mathcal{O}(0) e^{-\hat{H}\tau} \mathcal{O}(0) \rangle} = \int d\omega\, e^{-\omega\tau}\, \underset{\text{want}}{\langle \mathcal{O}(0) \delta(\omega - \hat{H}) \mathcal{O}(0) \rangle}$$

$$\underset{\text{have}}{G(\tau)} = \int d\omega\, e^{-\omega\tau}\, \underset{\text{want}}{\rho(\omega)}$$

- ***Linear, model-independent reconstruction*** (e.g. Backus-Gilbert-like, Chebyshev)

$$\sum_\tau \mathcal{K}(\bar{\omega},\tau) G(\tau) = \sum_\tau \mathcal{K}(\bar{\omega},\tau) \int d\omega\, e^{-\omega\tau}\, \rho(\omega) = \int d\omega \left[\sum_\tau \mathcal{K}(\bar{\omega},\tau)\, e^{-\omega\tau} \right] \rho(\omega)$$

$$= \int d\omega\, \widehat{\delta}_\Delta(\bar{\omega},\omega)\, \rho(\omega)$$

δ is exactly known

- Non-linear (***not discussed here…***)
  - Maximum Entropy Method (MEM)
  - Direct fits
  - Neural networks

See multiple ECT and CERN workshops, work by Aarts, Allton, Amato, Brandt, Burnier, Del Debbio, Francis, Giudice, Hands, Harris, Hashimoto, Jäger, Karpie, Liu, Meyer, Monahan, Orginos, Robaina, Rothkopf, Ryan, …*

- Key idea here… we aim only to construct $\widehat{\rho}(\bar{\omega}) \equiv \int_{-\infty}^{\infty} d\omega\, \widehat{\delta}_\Delta(\bar{\omega},\omega)\, \rho(\omega)$

Linear reconstruction (2/2)

$$\widehat{\rho}(\bar{\omega}) = \sum_{\tau} \mathcal{K}(\bar{\omega}, \tau) G(\tau) = \int_{-\infty}^{\infty} d\omega \, \widehat{\delta}_{\Delta}(\bar{\omega}, \omega) \rho(\omega)$$

$$\widehat{\delta}_{\Delta}(\bar{\omega}, \omega) = \sum_{\tau} \mathcal{K}(\bar{\omega}, \tau) e^{-\omega\tau}$$

- Given $\delta^{\text{target}}(\bar{\omega}, \omega)$, best $K(\bar{\omega}, \tau)$ = whatever minimizes combination of

$$\Delta(\mathcal{K} \,|\, \bar{\omega}, \omega) = \left| \delta^{\text{target}}(\bar{\omega}, \omega) - \sum_{\tau} \mathcal{K}(\bar{\omega}, \tau) e^{-\omega\tau} \right|$$

\+

statistical uncertainty on $\widehat{\rho}(\omega)$

- $\Delta(K \,|\, \bar{\omega}, \omega)$ is known
- If $\Delta(K_A \,|\, \bar{\omega}, \omega) < \Delta(K_B \,|\, \bar{\omega}, \omega)$ (for similar uncertainties) … then K_A is just better
- Chebyshev polynomials and Backus-Gilbert-like approaches are some options

Chebyshev idea

- Take $\bar{\omega}$ and Δ and 'target shape' as fixed and write…

$$\delta_\Delta^{\text{target}}(\bar{\omega},\omega) = f(\omega) = \sum_n \mathcal{K}_n (e^{-\omega a_t})^n = \sum_n \mathcal{K}_n x^n$$

- One specific family of choices for K_n is given using Chebyshevs

$$f(\omega) = \sum_n^N c_n T_n(x)$$

can be readily converted to K_n and compared to other methods

- Any method is best if and only if it achieves combined minimization of

$$\boxed{\Delta(\mathcal{K}\,|\,\bar{\omega},\omega) = \left| \delta^{\text{target}}(\bar{\omega},\omega) - \sum_\tau \mathcal{K}(\bar{\omega},\tau) e^{-\omega\tau} \right|}$$

+

statistical uncertainty on $\widehat{\rho}(\omega)$

First examples

we want…

$$\widehat{\rho}(\bar{\omega}) \equiv \int_0^{\infty} d\omega\, \widehat{\delta}_{\Delta}(\bar{\omega},\omega)\,\rho(\omega)$$

we have…

$$G(\tau) = \int d\omega\, e^{-\omega\tau}\,\rho(\omega)$$

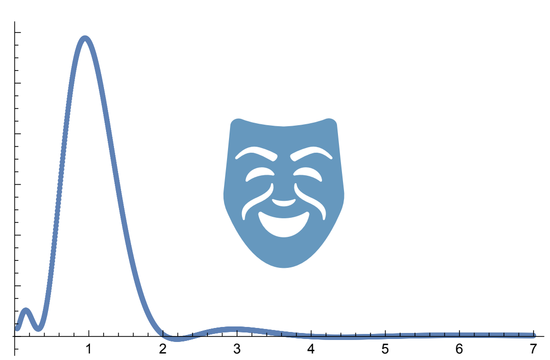
we design coefficients

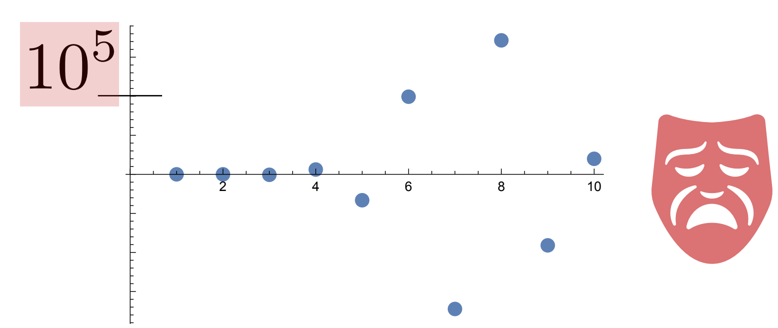
$$\sum_{\tau} \mathcal{K}(\bar{\omega},\tau)\, e^{-\omega\tau} = \widehat{\delta}_{\Delta}(\bar{\omega},\omega)$$

to estimate

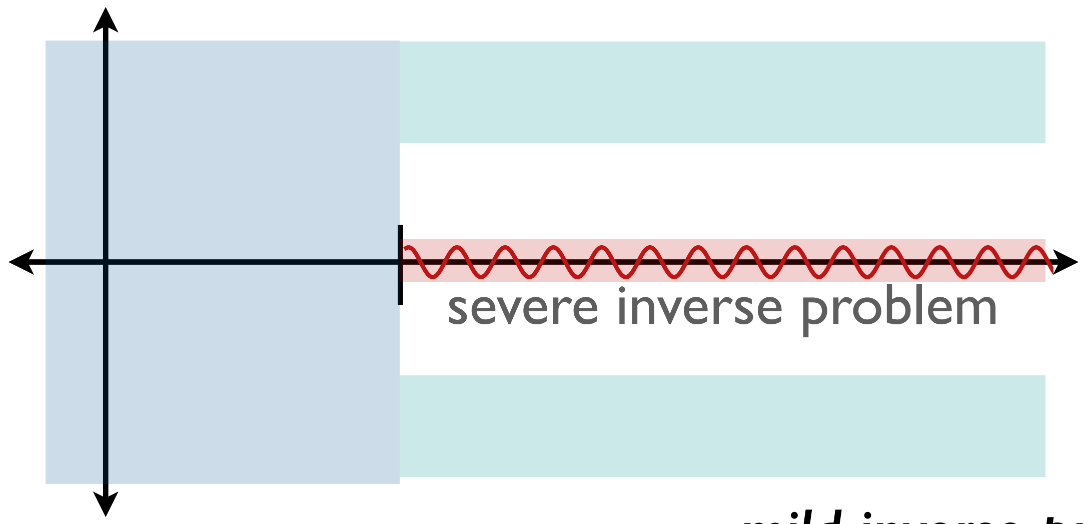
$$\widehat{\rho}(\bar{\omega}) = \sum_{\tau} \mathcal{K}(\bar{\omega},\tau)\, G(\tau)$$

$\widehat{\delta}_{\Delta}(\bar{\omega},\omega)$	$\mathcal{K}(\bar{\omega},\tau)$	
$e^{-\omega/\bar{\omega}}$	$\delta(\bar{\omega}-1/\tau)$	
$\dfrac{1}{\bar{\omega}-\omega}$	$e^{\bar{\omega}\tau}$	only below threshold





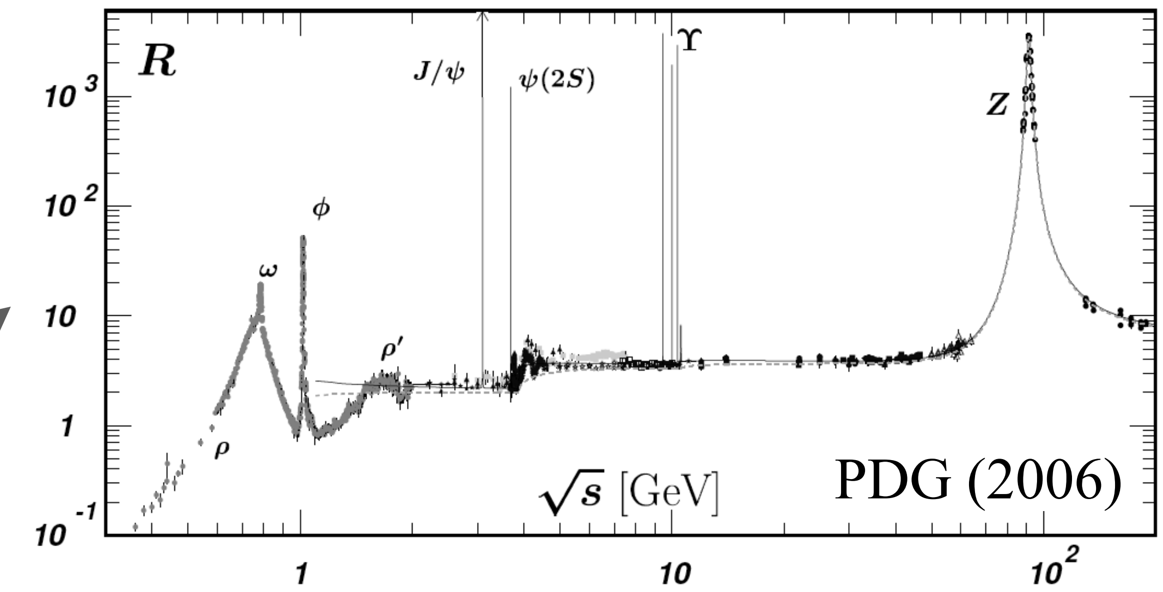
R ratio



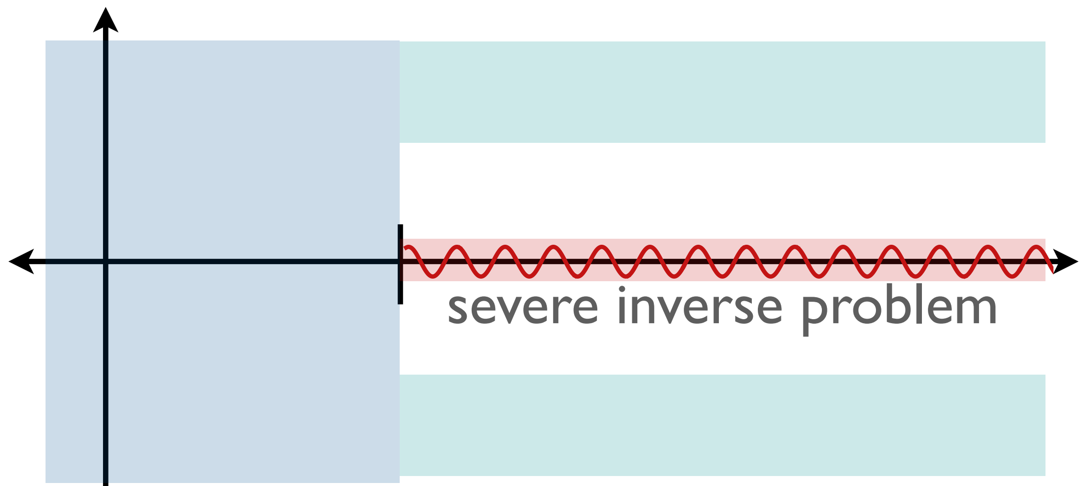
no inverse problem

mild inverse problem?

Hashimoto, 2019 $\frac{1}{\bar{\omega}-\omega}$



R ratio



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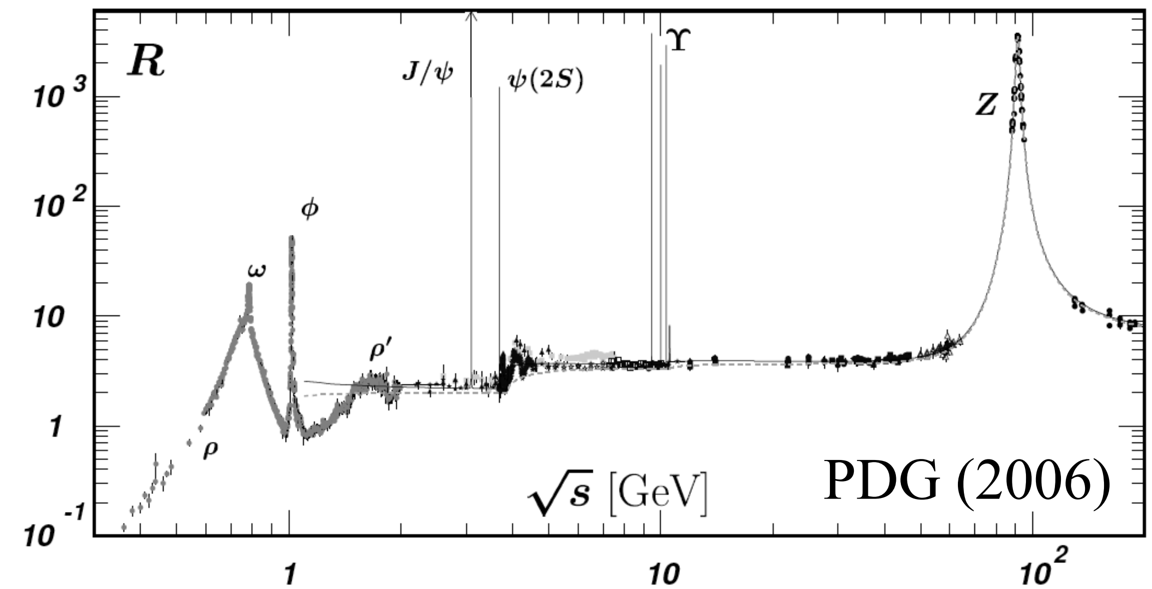


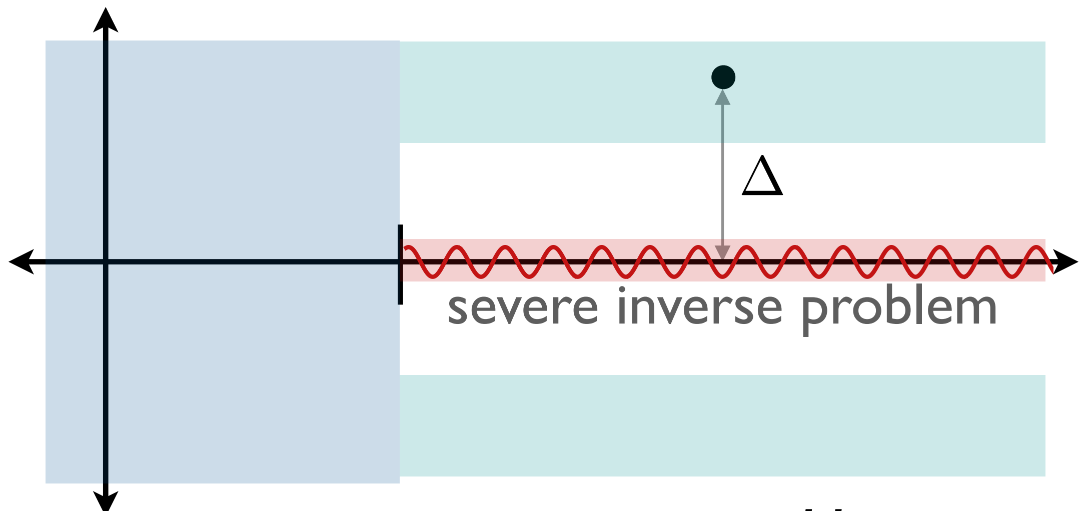
PHYSICAL REVIEW D VOLUME 13, NUMBER 7 1 APRIL 1976

Smearing method in the quark model*

E. C. Poggio, H. R. Quinn,† and S. Weinberg
Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138
(Received 8 December 1975)

We propose that perturbation theory in a gauge field theory of quarks and gluons can be used to calculate physical mass-shell processes, provided that both the data and the theory are smeared over a suitable energy range. This procedure is explored in detail for the process of electron-positron annihilation into hadrons. We show that a smearing range of 3 GeV^2 in the squared center-of-mass energy should be adequate to allow the use of lowest-order perturbation theory. The smeared data are compared with theory for a variety of models. It appears to be difficult to fit the present data with theory, unless there is a new charged lepton or quark with mass in the region 2 to 3 GeV.

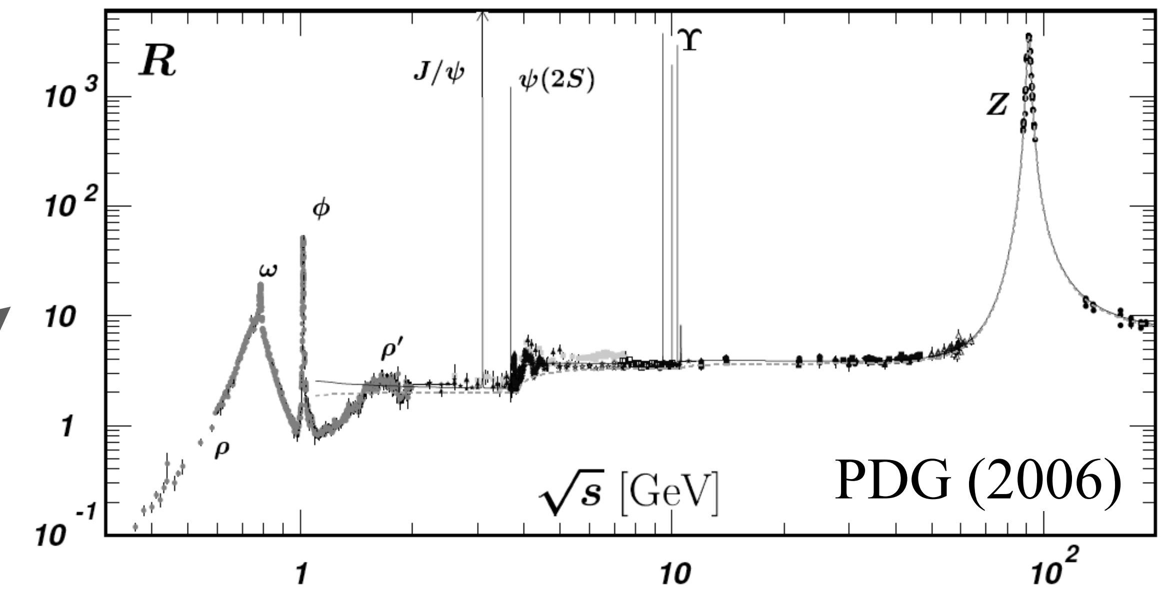
R ratio

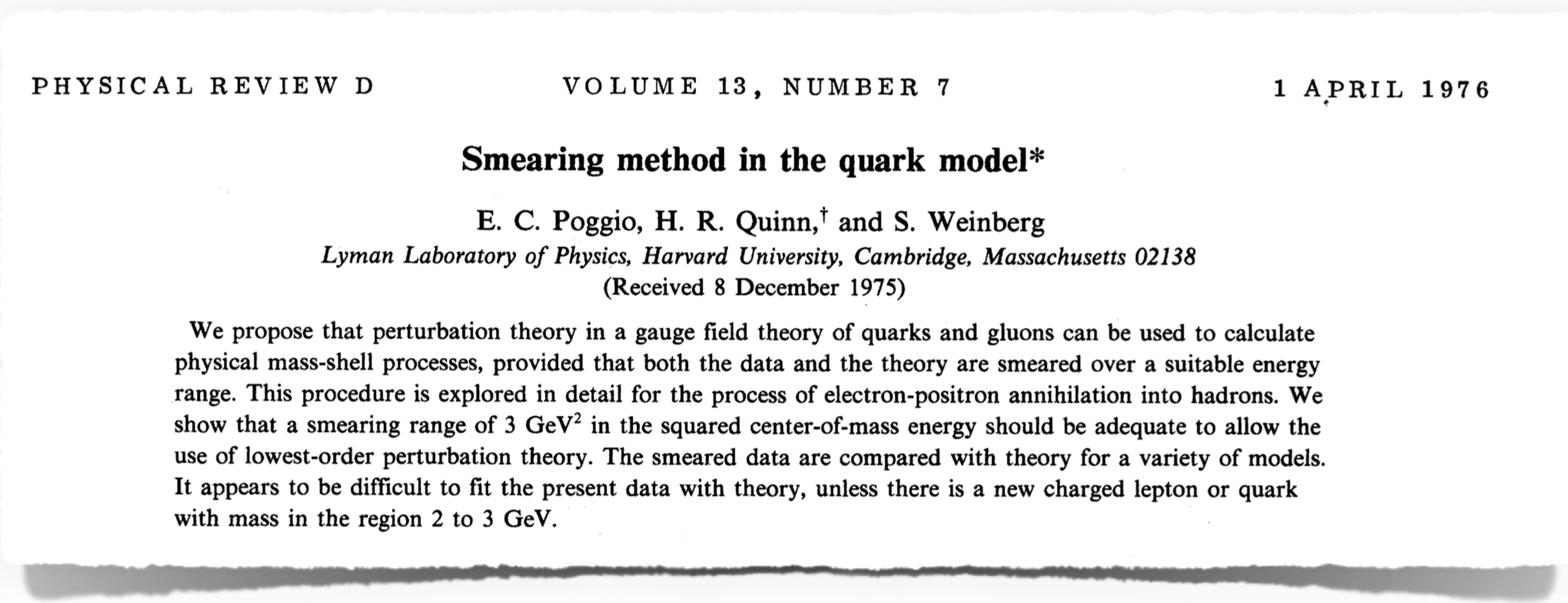


no inverse problem

Hashimoto, 2019 $\frac{1}{\bar{\omega} - \omega}$

mild inverse problem?





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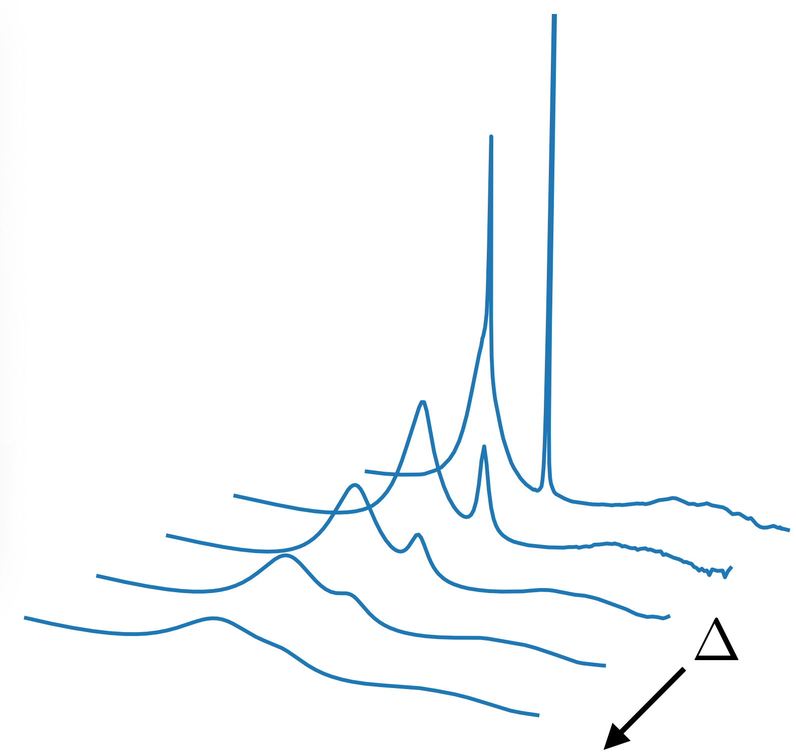
$$\widehat{R}_\Delta(s) = \frac{\Delta}{\pi} \int_0^\infty ds' \, \frac{R(s')}{(s'-s)^2 + \Delta^2}$$

into the complex plane!



courtesy of M. Bruno
(using F. Jegerlehner's ***alphaQED***)

Miscellaneous comments

- If correlator is indistinguishable from a single exponential, then

$$G(\tau) = c_0 e^{-E_0 \tau} \quad \Longrightarrow \quad \rho(\omega) = c_0 \delta(\omega - E_0)$$

- Ground-state-dominated time slices do not add much

- Systematic uncertainty on $\widehat{\rho}(\bar{\omega})$ can be challenging

$$\Delta(\mathcal{K} \,|\, \bar{\omega}, \omega) = \left| \delta^{\text{target}}(\bar{\omega}, \omega) - \sum_{\tau} \mathcal{K}(\bar{\omega}, \tau) e^{-\omega \tau} \right|$$

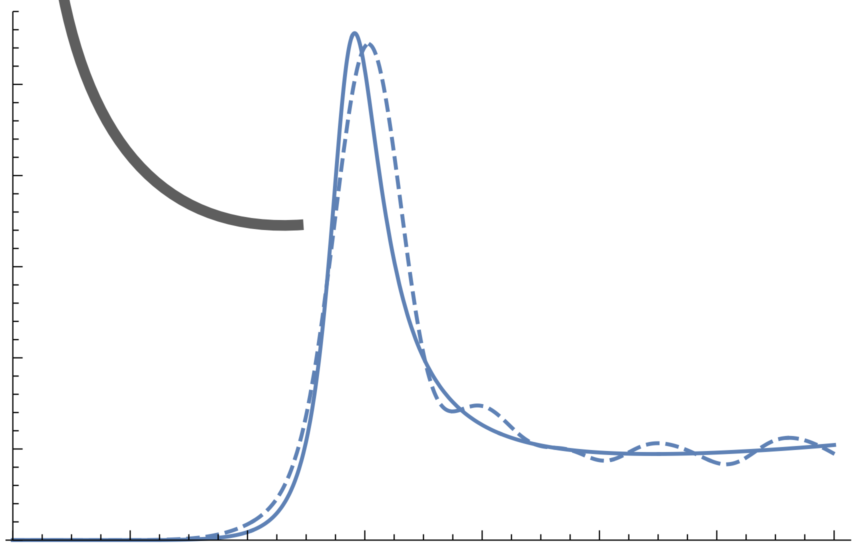
want $\Delta(K \,|\, \bar{\omega}, \omega)$ small when $\rho(\omega)$ is large

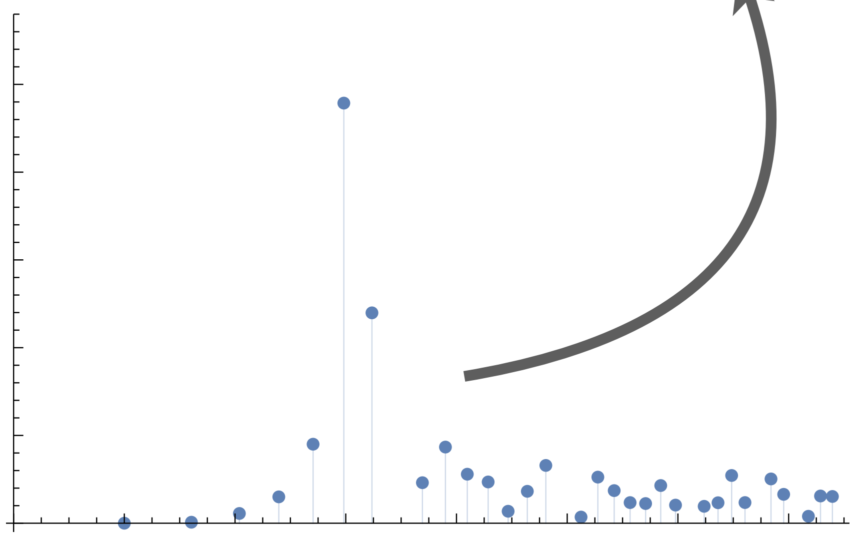
- A GEVP that saturates the correlator contains all relevant information

$$G(\tau) = \sum_n c_n e^{-E_n \tau} \quad \Longrightarrow \quad \rho(\omega) = \sum_n c_n \delta(\omega - E_n)$$

Role of the finite volume

$$\widehat{\rho}_L(\bar{\omega}) \equiv \int_0^\infty d\omega\, \widehat{\delta}_\Delta(\bar{\omega}, \omega)\, \rho_L(\omega) \qquad G_L(\tau) = \int d\omega\, e^{-\omega\tau}\, \rho_L(\omega)$$





- Any reconstructed spectral function that ≠ forest of deltas…

contains implicit smearing (or else $L \to \infty$)

We require…

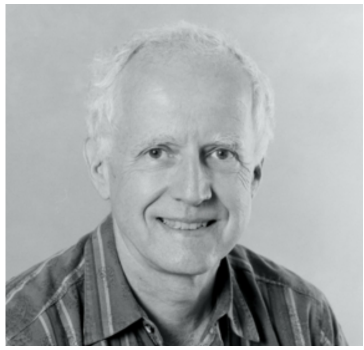
$$1/L \ll \Delta \ll \mu_{\text{physical}}$$

smearing function covers many delta peaks

smearing does not overly distort observable

MTH, Meyer, Robaina (2017)

Backus-Gilbert (*Numerical Recipes* version)




- Un-stabilized inverse

Without uncertainties, minimize the functional…

$$A[\mathcal{K}] \equiv \int_0^\infty d\omega\, (\bar{\omega} - \omega)^2\, \widehat{\delta}_\Delta(\bar{\omega}, \omega)^2 = \int_0^\infty d\omega\, (\bar{\omega} - \omega)^2 \Big[\sum_\tau \mathcal{K}_\tau e^{-\omega\tau} \Big]^2$$

with unit area constraint on δ

- Stabilized inverse

With uncertainties, instead minimize…

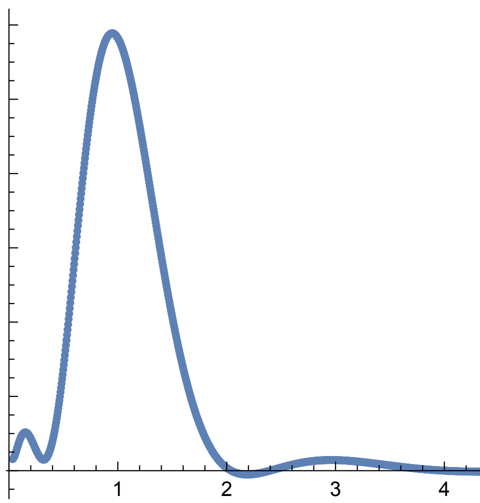
$$W_\lambda[\mathcal{K}] \equiv (1 - \lambda) A[\mathcal{K}] + \lambda \frac{\mathcal{K} \cdot \mathrm{COV} \cdot \mathcal{K}}{\mathcal{N}}$$

wants oscillating K *wants well-behaved K*

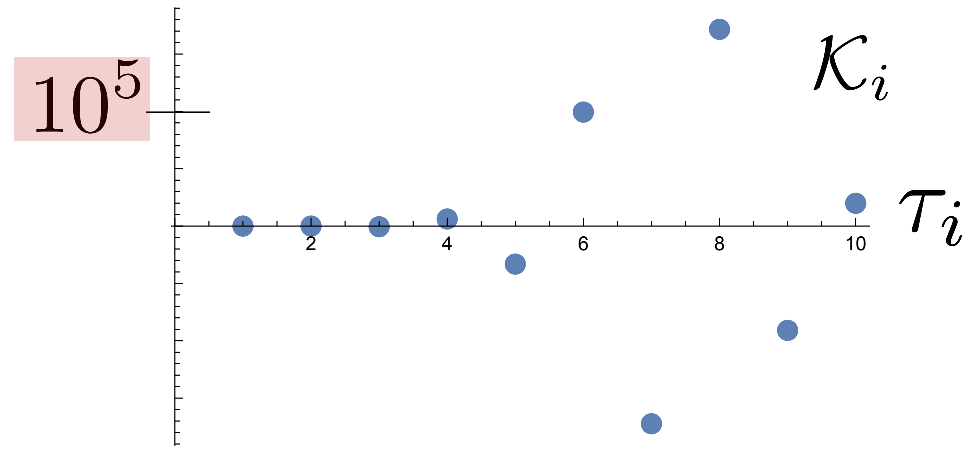
λ parametrizes the tradeoff between *final resolution* and *final* uncertainty

Backus-Gilbert example

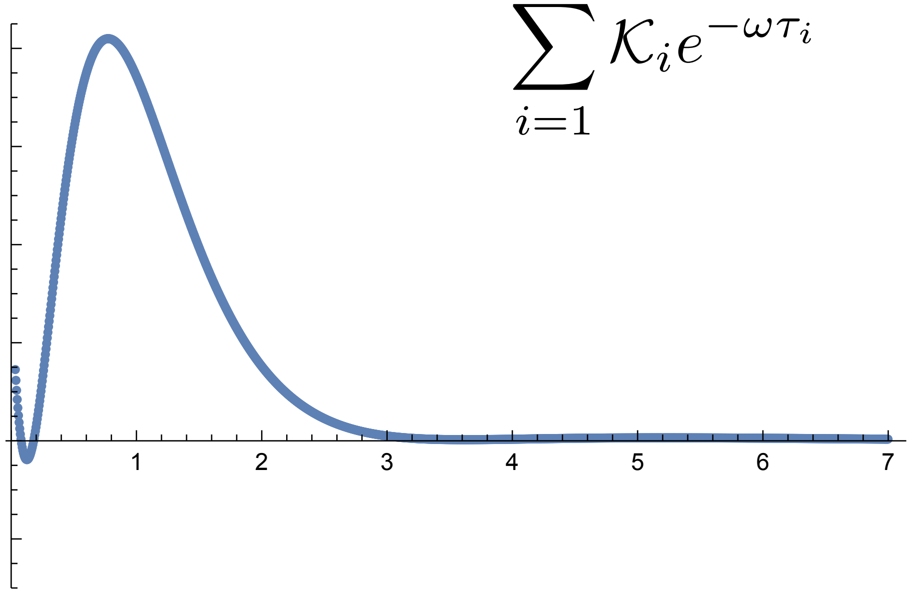
- Un-stabilized inverse



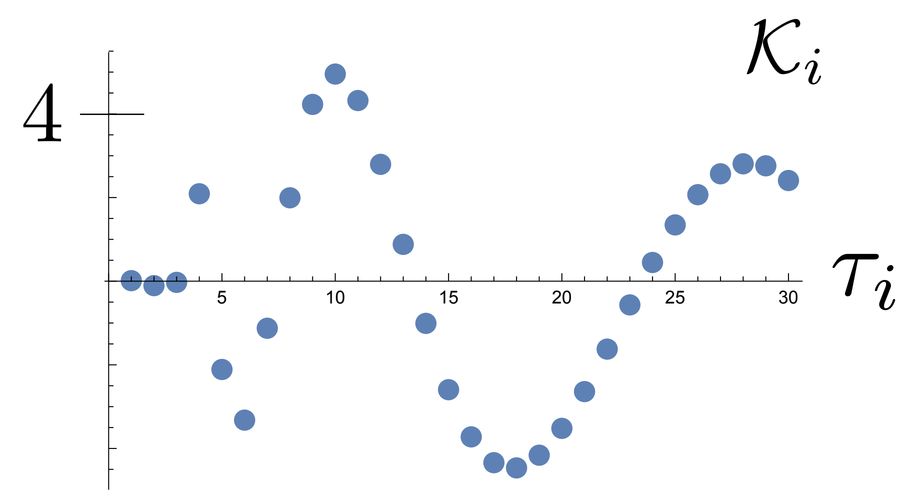
$$\sum_{i=1}^{10} \mathcal{K}_i e^{-\omega \tau_i}$$



- Stabilized inverse

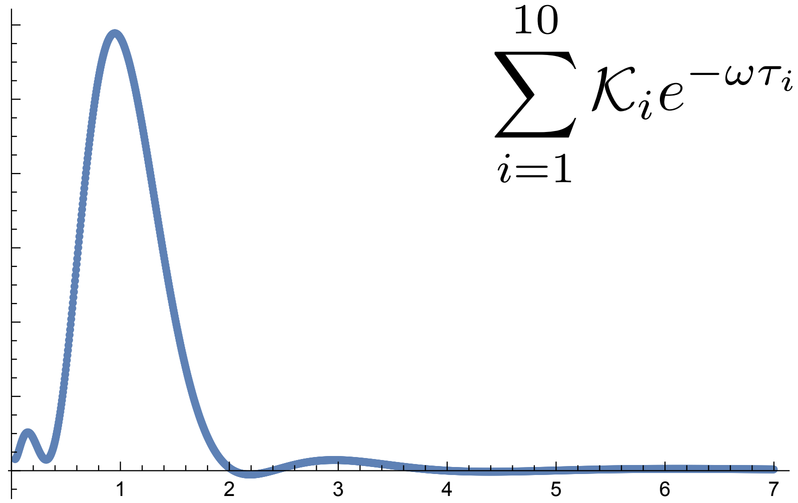


$$\sum_{i=1}^{30} \mathcal{K}_i e^{-\omega \tau_i}$$

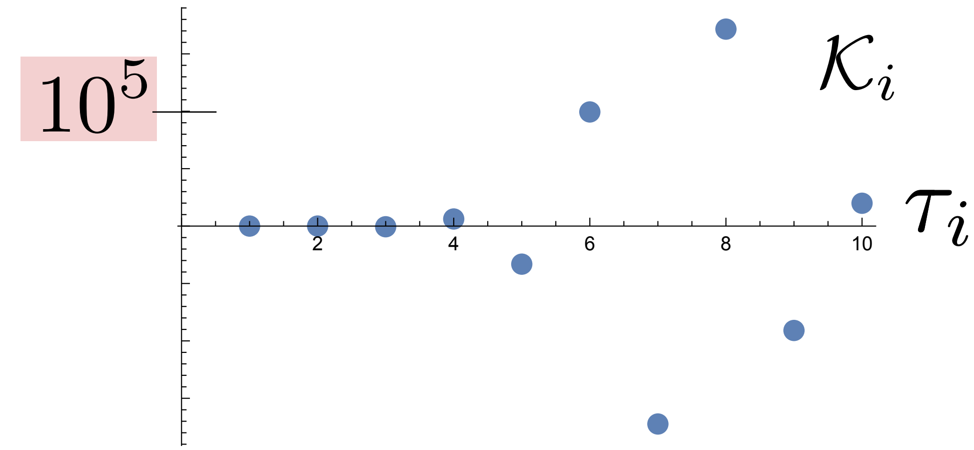


Backus-Gilbert example

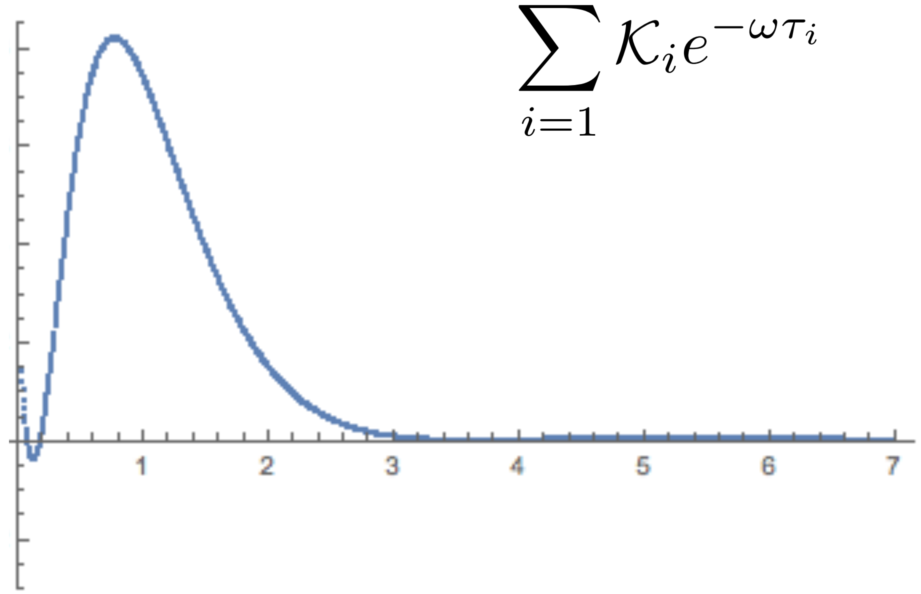
- Un-stabilized inverse



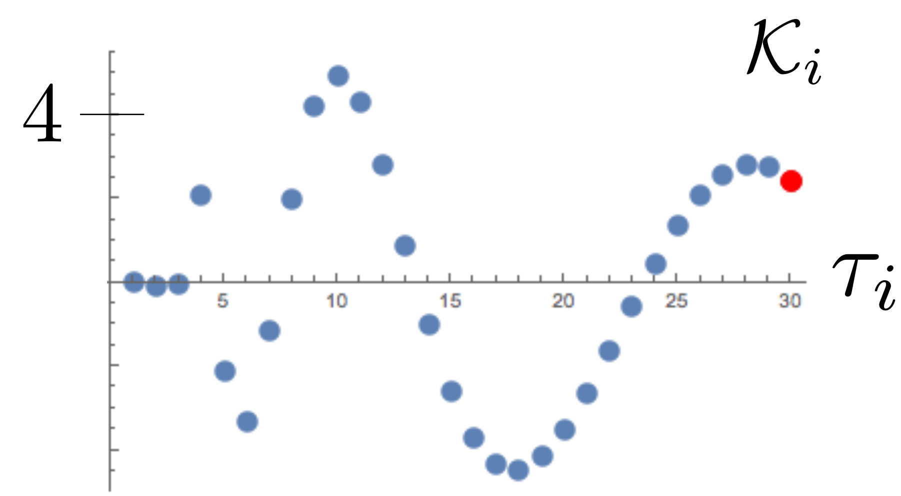
$$\sum_{i=1}^{10} \mathcal{K}_i e^{-\omega \tau_i}$$



- Stabilized inverse



$$\sum_{i=1}^{\bullet} \mathcal{K}_i e^{-\omega \tau_i}$$



Extended Backus Gilbert

- Important generalization from ***Martin Hansen, Lupo, Tantalo***
- Addressed two limitations…

Shape and width of δ depends on data quality

No clear way to set λ

$$W_\lambda[\mathcal{K}] \equiv (1-\lambda)A[\mathcal{K}] + \lambda \frac{\mathcal{K}\cdot \mathrm{COV}\cdot \mathcal{K}}{\mathcal{N}}$$

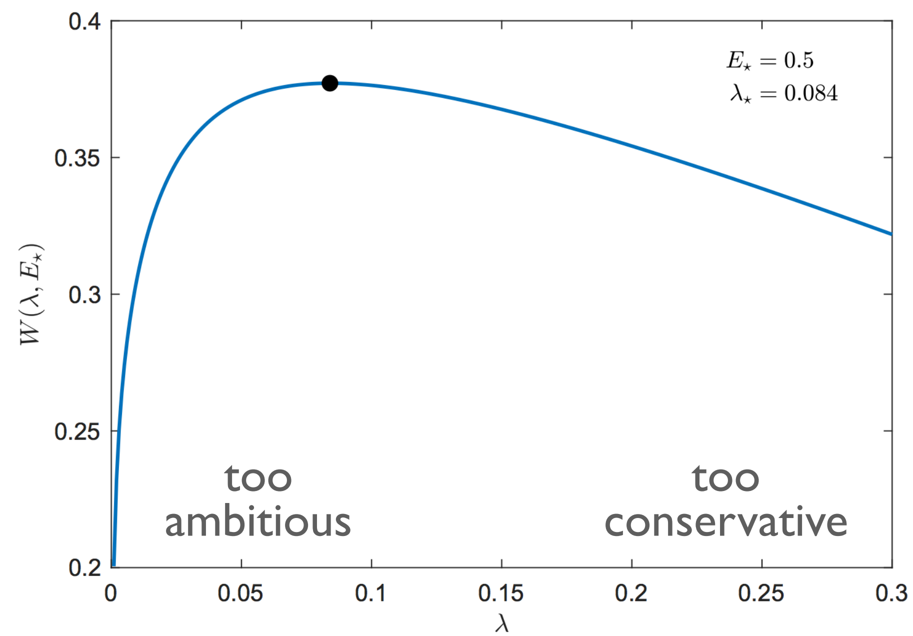
Change minimizing functional

$$A[\mathcal{K}] \equiv \int_0^\infty d\omega\,(\bar\omega-\omega)^2\,\widehat{\delta}_\Delta(\bar\omega,\omega)^2$$

$$\widetilde{A}[\mathcal{K}] \equiv \int_0^\infty d\omega \left[\widehat{\delta}_\Delta^{\,\mathrm{target}}(\bar\omega,\omega) - \widehat{\delta}_\Delta(\bar\omega,\omega)\right]^2$$

Extremize *W w.r.t. λ*

$$W_\lambda[\mathcal{K}_{\min}(\lambda)]$$

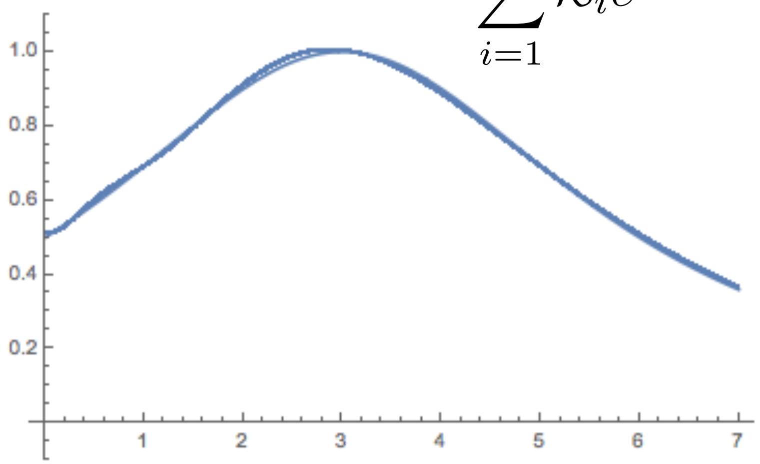


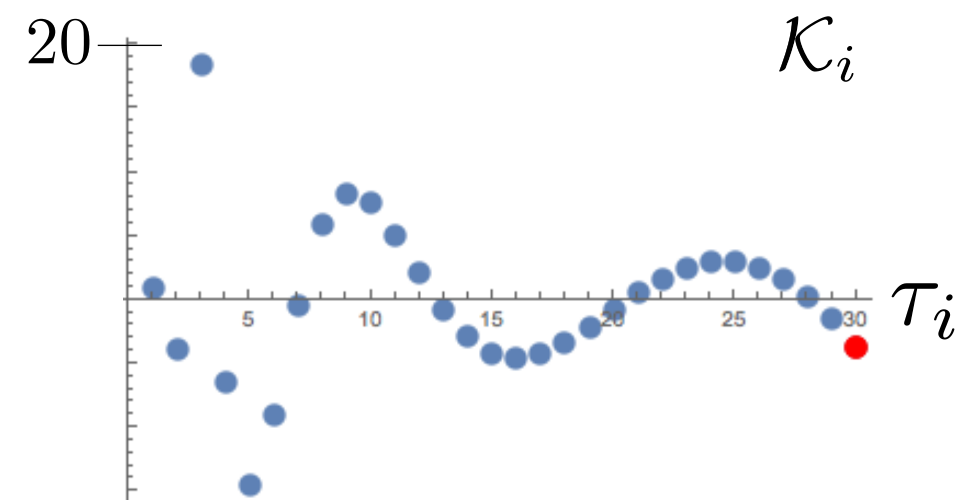
Hansen, Lupo, Tantalo (2019)

Extended Backus Gilbert example

- e.g. target a Breit-Wigner

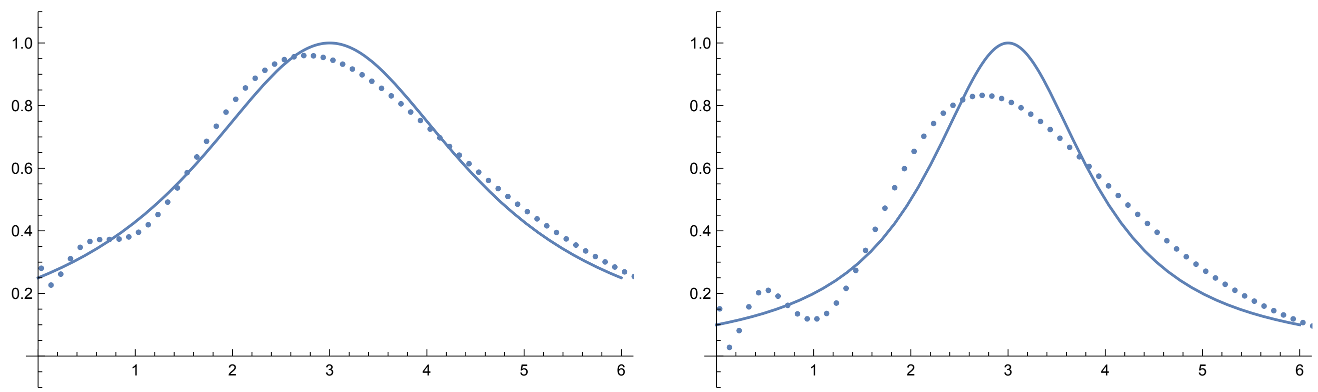
$$\sum_{i=1} \mathcal{K}_i e^{-\omega \tau_i}$$



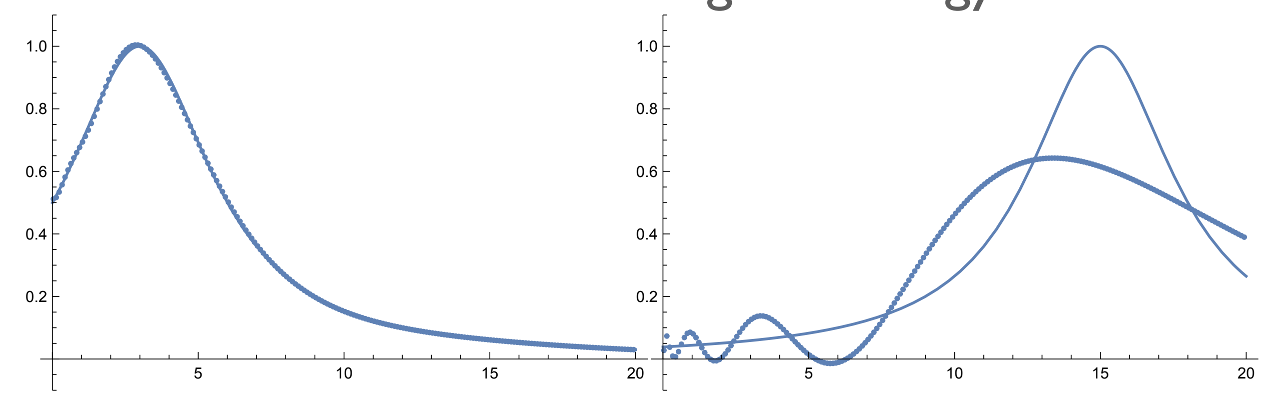


- Method will fail if one tries to…

make the resolution function too narrow



move the resolution too high in energy

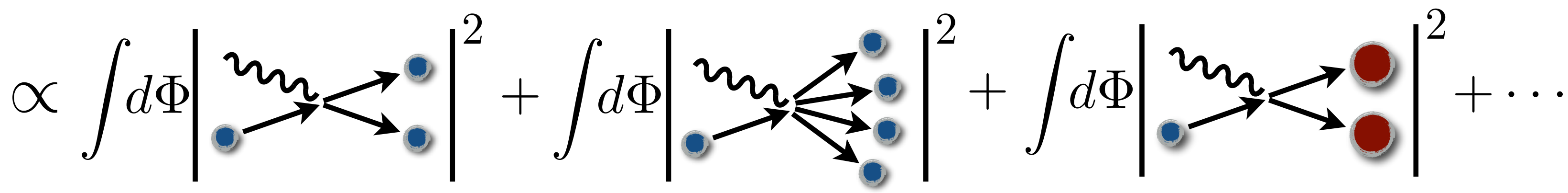


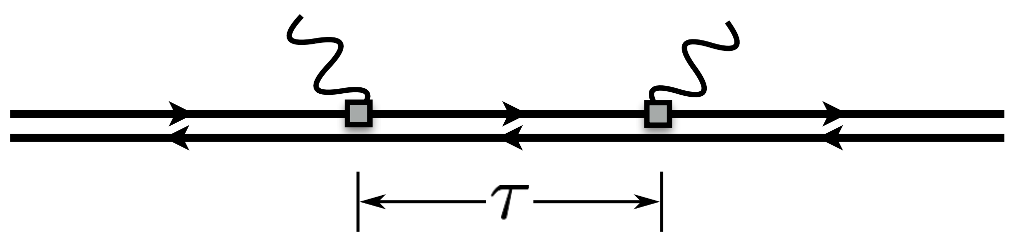
Hansen, Lupo, Tantalo (2019)

Total rate based applications

- Hadronic tensor, heavy flavor lifetimes

$$W_{\mu\nu}(p,q) \equiv \int d^4x \; e^{iqx} \langle \pi, p | \mathcal{J}_\mu^\dagger(x) \mathcal{J}_\nu(0) | \pi, p \rangle \qquad \propto \langle \pi, p | \widetilde{\mathcal{J}}^\dagger_{\boldsymbol{q},\mu}(0) \delta(\hat{H} - \omega) \mathcal{J}_\nu(0) | \pi, p \rangle$$





$$= \int_0^\infty d\omega \, e^{-\omega\tau} \, W_{\mu\nu}(p,q)_{\omega = p^0 + q^0}$$

$$W_{\mu\nu} = \lim_{\Delta \to 0} \lim_{L \to \infty} \widehat{W}_{\mu\nu;\Delta,L}$$

- Inclusive and semi-inclusive, importance of smearing/volume/kinematics
- What about ***scattering*** and ***transition amplitudes?***

MTH, Meyer, Robaina (2017), see also K.F.-Liu (2016), Hashimoto (2019-2021)

Amplitudes from spectral functions

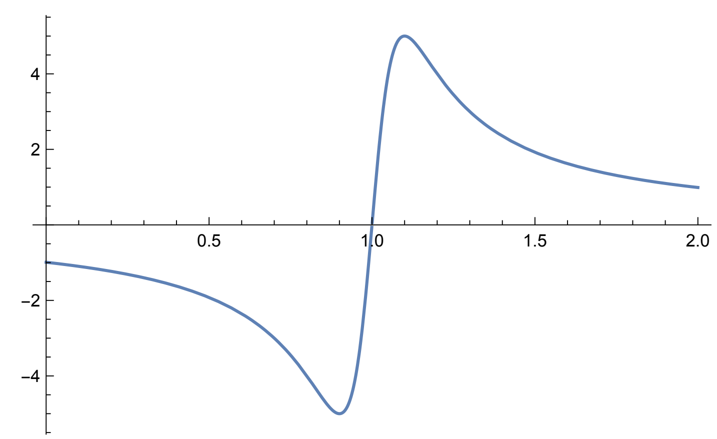
- First define the Euclidean correlator

$$G_L(\tau) = \langle \pi_{\boldsymbol{p}_4} | \pi(\tau_3, \boldsymbol{p}_3) \pi(0) | \pi_{\boldsymbol{p}_1} \rangle_L = \int_0^\infty dE_3 \, \rho(E_3) e^{-E_3 \tau_3}$$

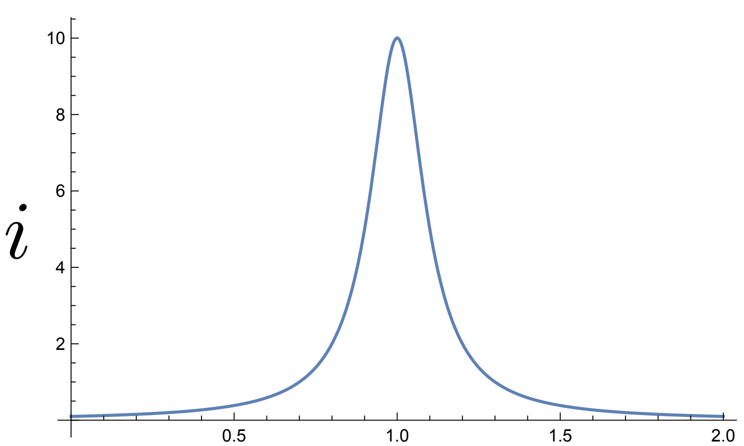
and the smeared spectral function

$$\widehat{\rho}^{L,\epsilon}_{\boldsymbol{p}_4 \boldsymbol{p}_1}(q_3) = \int_0^\infty dE_3 \, \frac{1}{q_3^0 - E_3 + i\epsilon} \, \rho(E_3) = \int_0^\infty dE_3 \, \widehat{\delta}_\epsilon(q_3^0, E_2) \, \rho(E_3)$$

$$\frac{1}{q_3^0 - E_3 + i\epsilon} =$$



$$-i$$



Bulava, MTH (2019)

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- Next project *on shell* at finite ϵ

$$\mathcal{M}_{\mathrm{c}}^{L,\epsilon}(p_4 p_3 | p_2 p_1) \equiv \frac{2E(\boldsymbol{p}_3)}{Z^{1/2}(\boldsymbol{p}_3)} \frac{2E(\boldsymbol{p}_2)}{Z^{1/2}(\boldsymbol{p}_2)} \epsilon^2 \, \widehat{\rho}^{L,\epsilon}_{\boldsymbol{p}_4 \boldsymbol{p}_1}(E(\boldsymbol{p}_3), \boldsymbol{p}_3)$$

- Finally project out the ***scattering amplitude***

$$\mathcal{M}_{\mathrm{c}}(p_4 p_3 | p_2 p_1) = \lim_{\epsilon \to 0^+} \lim_{L \to \infty} \mathcal{M}_{\mathrm{c}}^{L,\epsilon}(p_4 p_3 | p_2 p_1)$$

Bulava, MTH (2019)

Some comments

- Derivation based in modified **LSZ** + ***signature-independence*** of ρ(E)

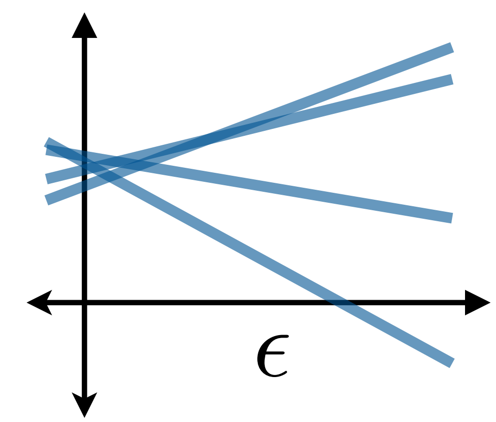
- Holds when LSZ holds

$$\langle m, \mathrm{out} | n, \mathrm{in} \rangle \qquad\qquad \langle m, \mathrm{out} | \mathcal{J}(0) | n, \mathrm{in} \rangle$$

- Some nice features…

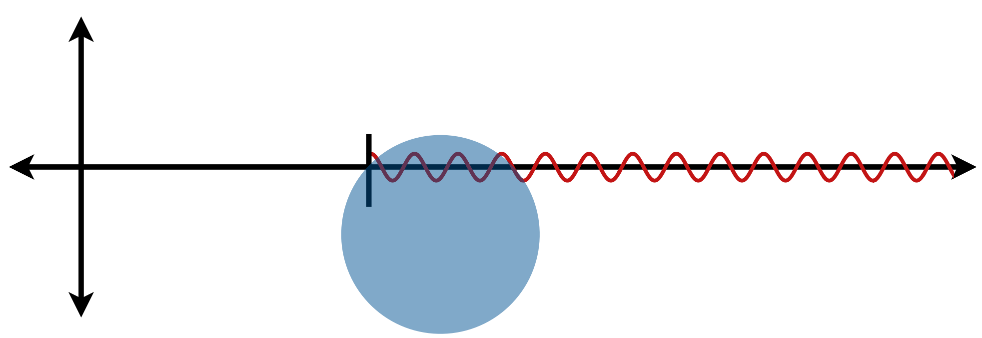
GEVP-like operator freedom

$$G_L^{[ab]}(\tau) = \langle \pi_{\boldsymbol{p}_4} | \pi^a(\tau_3, \boldsymbol{p}_3) \pi^b(0) | \pi_{\boldsymbol{p}_1} \rangle_L \longrightarrow \mathcal{M}^{[ab],\epsilon,L}$$



Finite range of analyticity in ϵ

Uses overlaps



Bulava, MTH (2019)

Perturbative study…

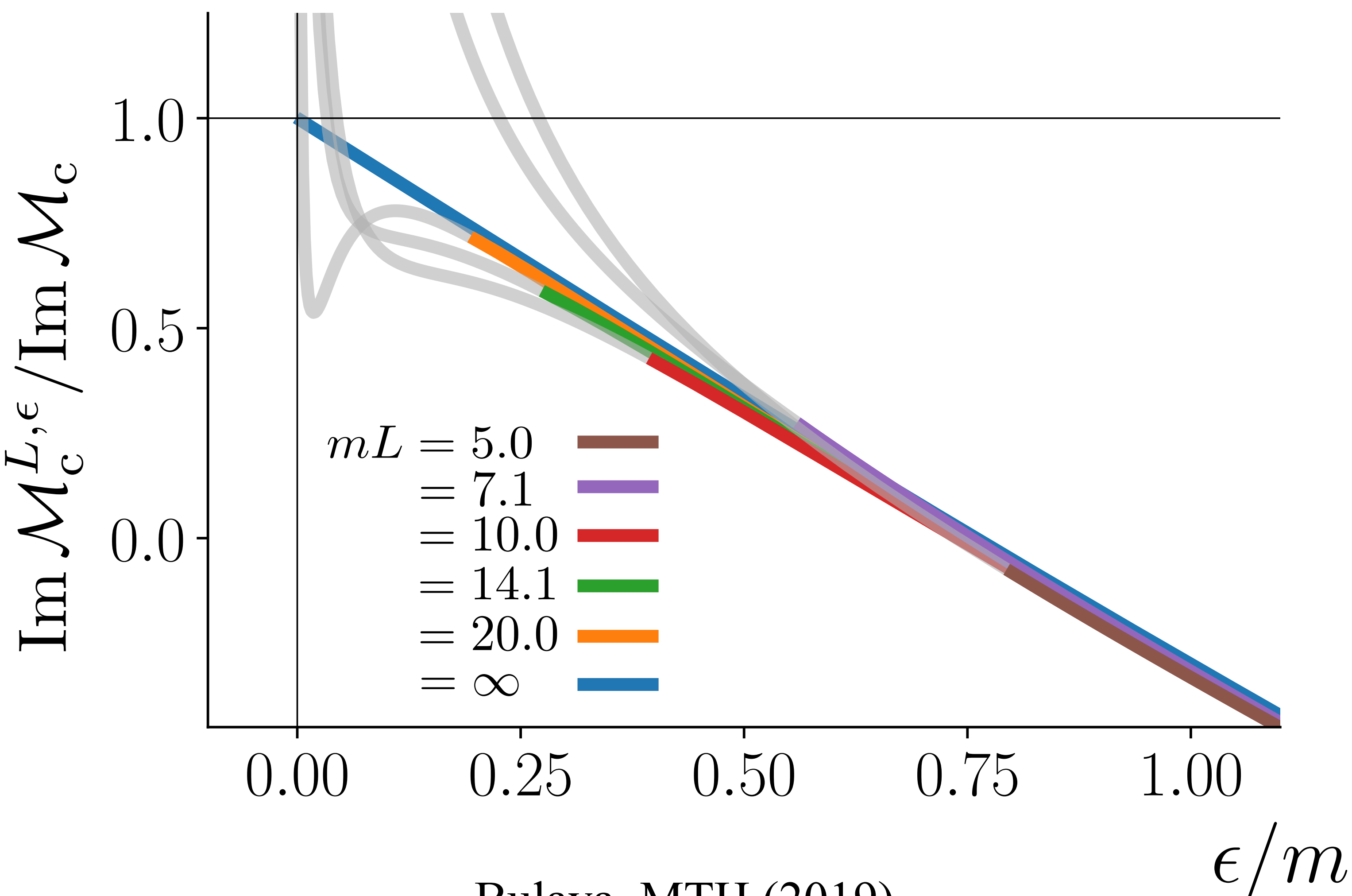
Calculate in PT

$$G_L(\tau) = \langle \pi_{\boldsymbol{p}_4} | \pi(\tau_3, \boldsymbol{p}_3) \pi(0) | \pi_{\boldsymbol{p}_1} \rangle_L$$

Convert to this

$$\mathcal{M}_{\mathrm{c}}^{L,\epsilon}(p_4 p_3 | p_2 p_1)$$

$$\mathrm{Im}\,\mathcal{M}_{\mathrm{c}}^{L,\epsilon}(p_4 p_3 | p_2 p_1) = \frac{\lambda^2}{2} \frac{1}{L^3} \sum_{\boldsymbol{k}'}^{\Lambda} \frac{1}{(2E(\boldsymbol{k}'))^2}\, \mathrm{Im} \left\{ \frac{1}{(E_{\mathrm{cm}} - 2E(\boldsymbol{k}') + i\epsilon)} \left[1 - \frac{\epsilon^2}{4E(\boldsymbol{k}')^2} - \frac{\epsilon(\epsilon + 2iE(\boldsymbol{k}'))}{E_{\mathrm{cm}} E(\boldsymbol{k}')} \right] \right\}$$



Bulava, MTH (2019)

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Convert to this

$$\mathcal{M}_{\rm c}^{L,\epsilon}(p_4 p_3 | p_2 p_1)$$

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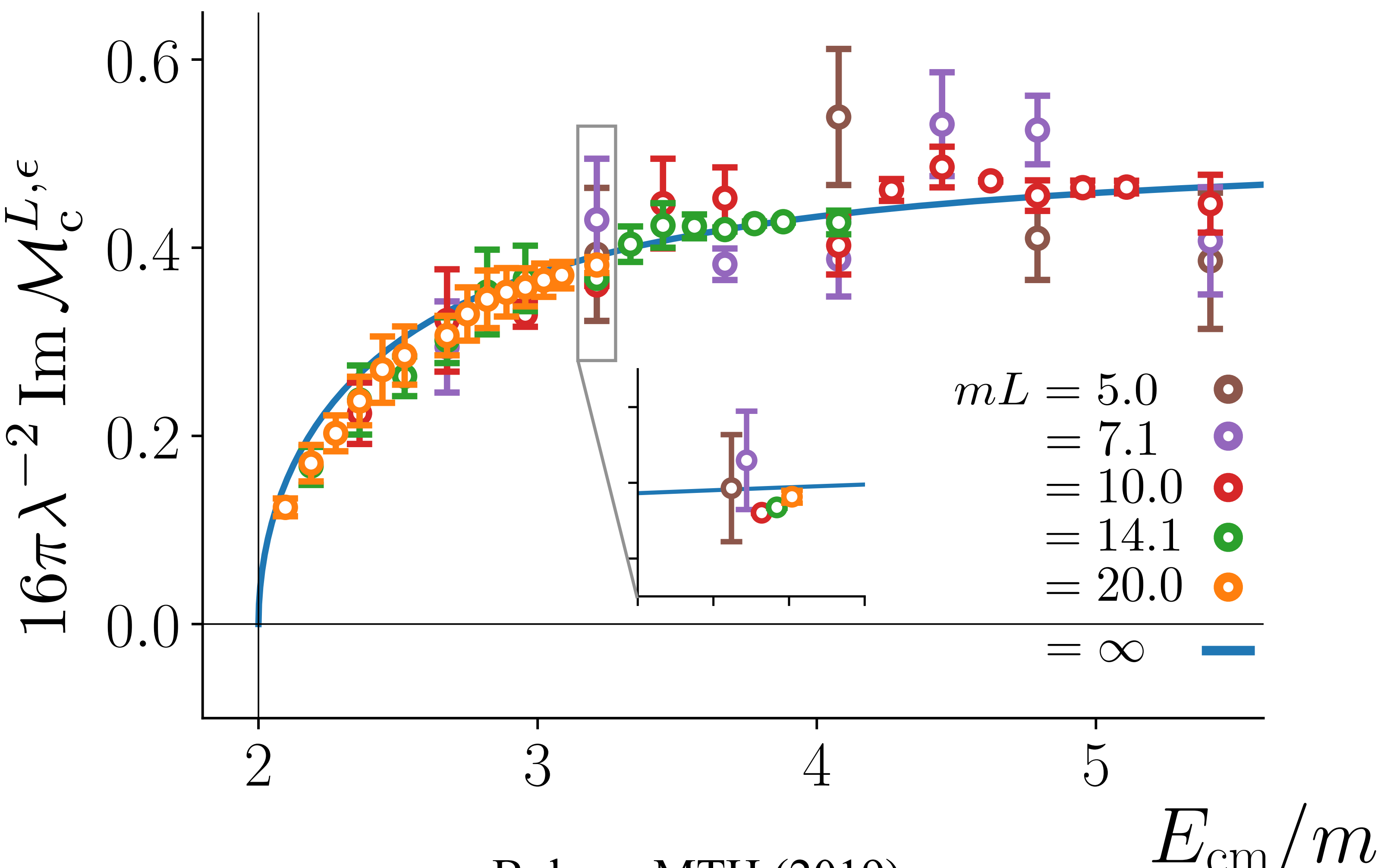


Bulava, MTH (2019)

Monte-Carlo test

- Full lattice calculation in two-dimensional O(3) non-linear sigma model

- Demonstrating the modified Backus-Gilbert (HLT) method for the "R-ratio"

$$C(t) \equiv \int d\boldsymbol{x} \, \langle \Omega | \, \hat{j}_1^a(0, \boldsymbol{x}) \, e^{-\hat{H}t} \, \hat{j}_1^a(0) \, | \Omega \rangle = \int_0^\infty d\omega \, e^{-\omega t} \, \rho(\omega)$$

- Data + theory driven analysis of finite-L and -T effects and discretization

ID	$(L/a) \times (T/a)$	β	$am_\star$	$m_\star L$	$m_\star T$
A1	640×320	1.63	0.0447989(62)	29	14
A2	1280×640	1.72	0.0257695(31)	33	17
A3	1920×960	1.78	0.0176104(31)	34	17
A4	2880×1440	1.85	0.0112608(29)	32	16
B1	5760×1440	1.85	0.0112607(73)	65	16
B2	2880×2880	1.85	0.0112462(72)	32	32

Bulava, MTH, Hansen, Patella, Tantalo (2021)

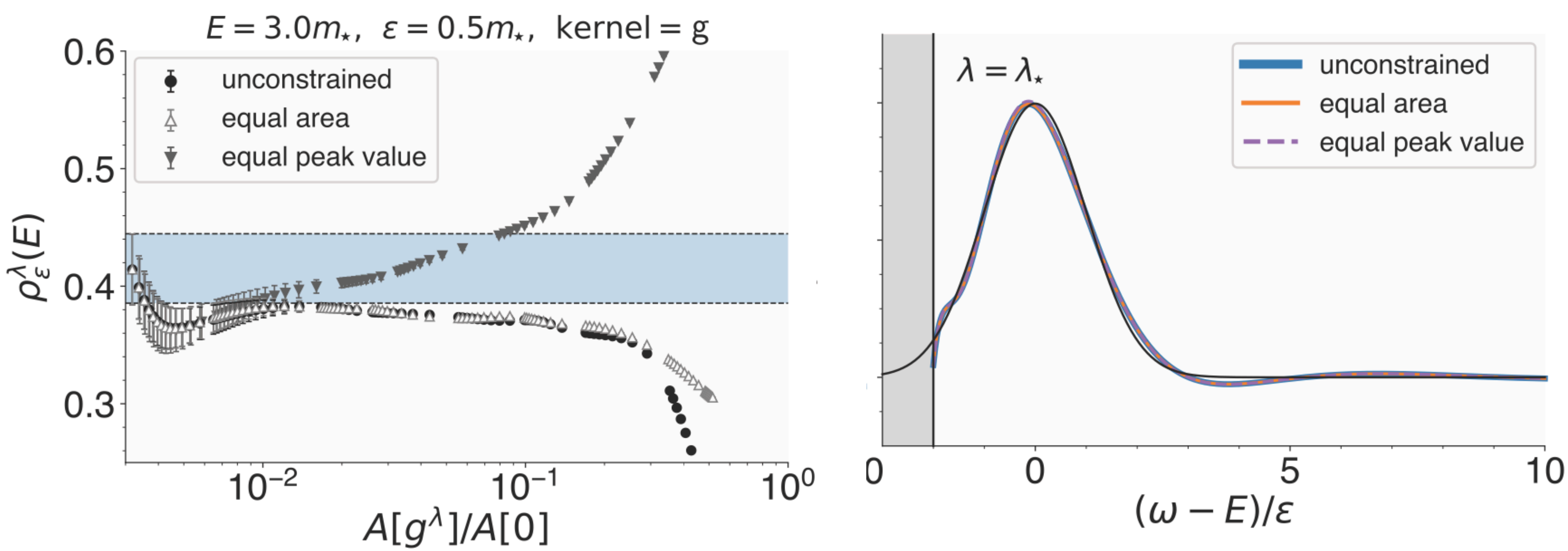
Monte-Carlo test

- Construct different smearings of $\rho(\omega)$

$$\rho^{\lambda}_{\epsilon}(E) = \int_0^{\infty} d\omega\, \delta^{\lambda}_{\epsilon}(E,\omega)\, \rho(\omega)$$

$$\delta^{\mathrm{g}}_{\epsilon}(x) = \frac{1}{\sqrt{2\pi}\epsilon} \exp\left[-\frac{x^2}{2\epsilon^2}\right], \qquad \delta^{\mathrm{c0}}_{\epsilon}(x) = \frac{1}{\pi}\frac{\epsilon}{x^2+\epsilon^2},$$

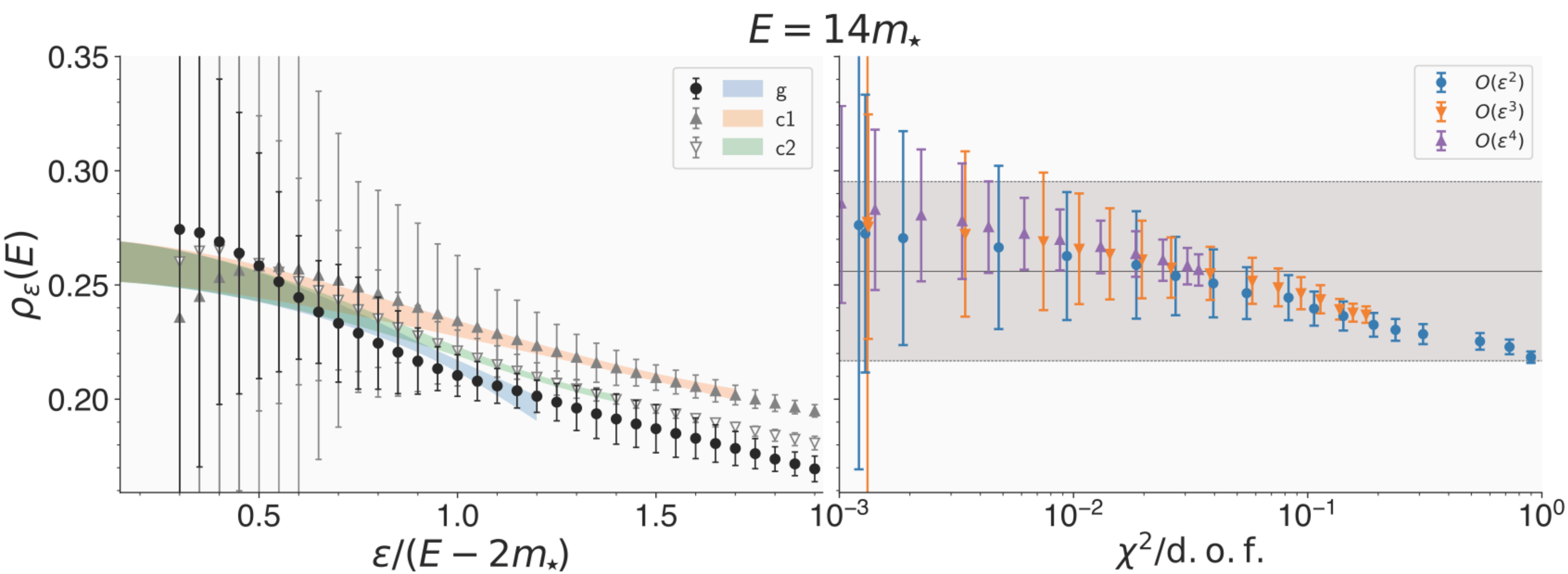
$$\delta^{\mathrm{c1}}_{\epsilon}(x) = \frac{2}{\pi}\frac{\epsilon^3}{(x^2+\epsilon^2)^2}, \qquad \delta^{\mathrm{c2}}_{\epsilon}(x) = \frac{8}{3\pi}\frac{\epsilon^5}{(x^2+\epsilon^2)^3}.$$



Bulava, MTH, Hansen, Patella, Tantalo (2021)

Extrapolation

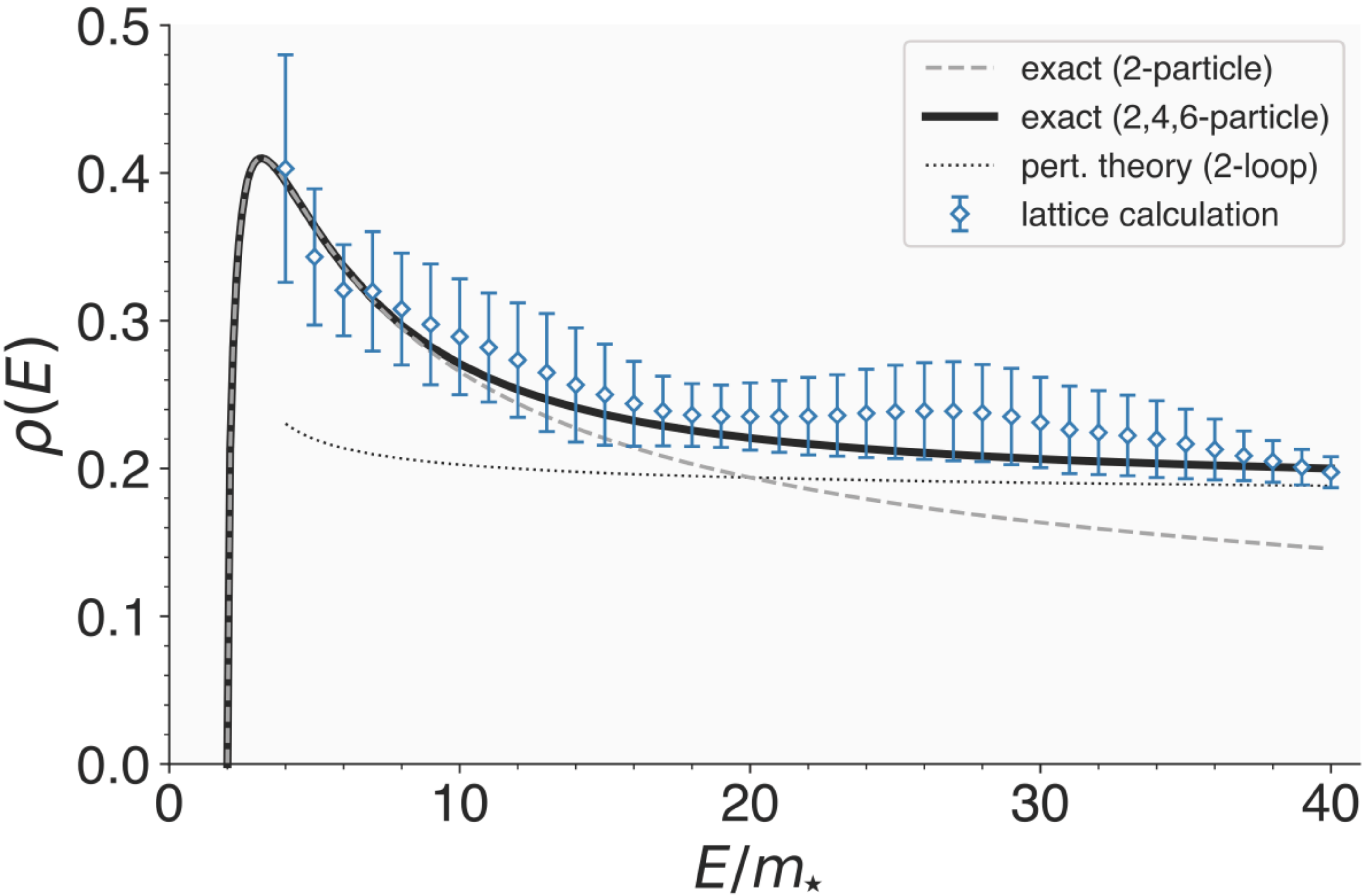
- Targeting $\rho(E)$ for $E = 14m$ here



- Use known relations between different smearing kernels

Bulava, MTH, Hansen, Patella, Tantalo (2021)

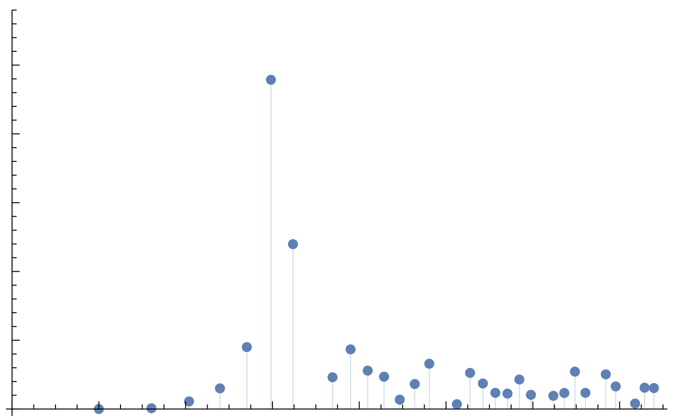
Result



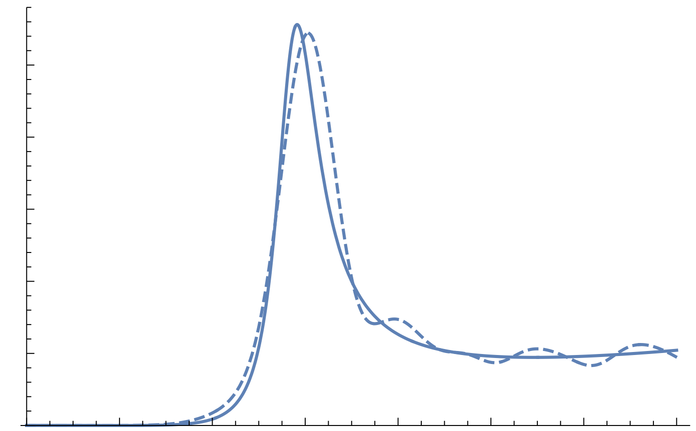
Bulava, MTH, Hansen, Patella, Tantalo (2021)

Spectral summary

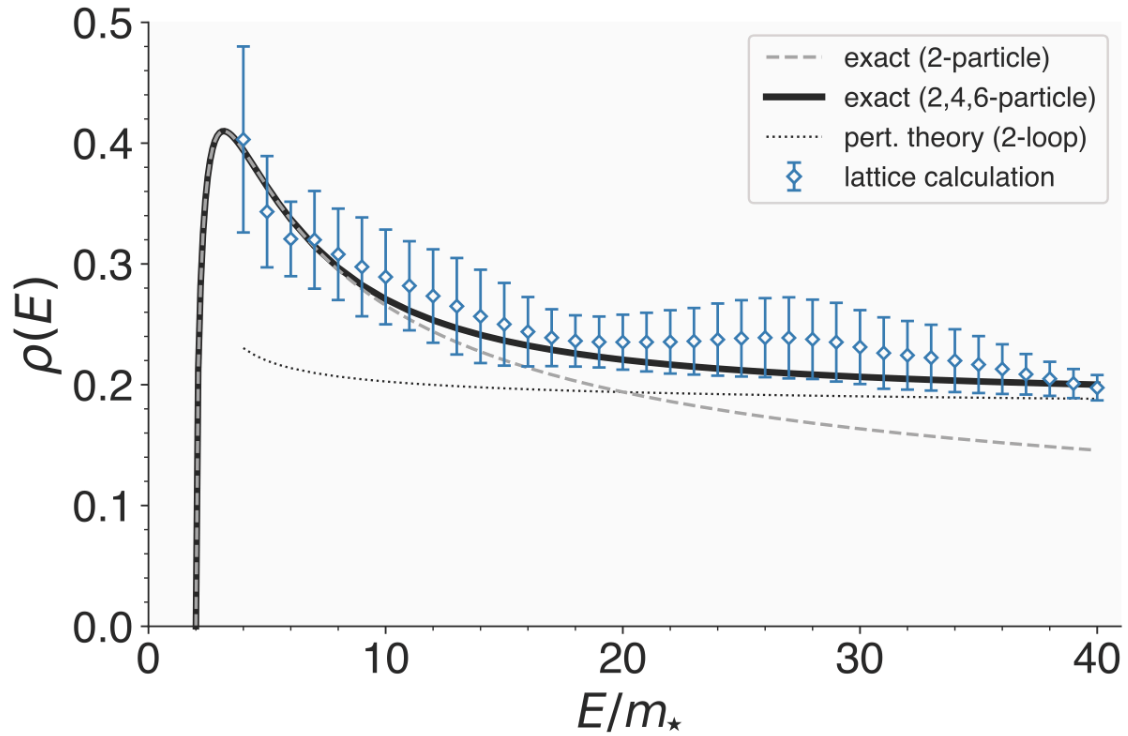
- Cannot solve the inverse problem, we can get $\widehat{\rho}_{L,\Delta}(\bar{\omega}) \equiv \int_0^\infty d\omega\, \widehat{\delta}_\Delta(\bar{\omega},\omega)\, \rho_L(\omega)$
- Smearing is needed anyway to ***suppress volume effects***



$$1/L \ll \Delta \ll \mu_{\text{physical}}$$



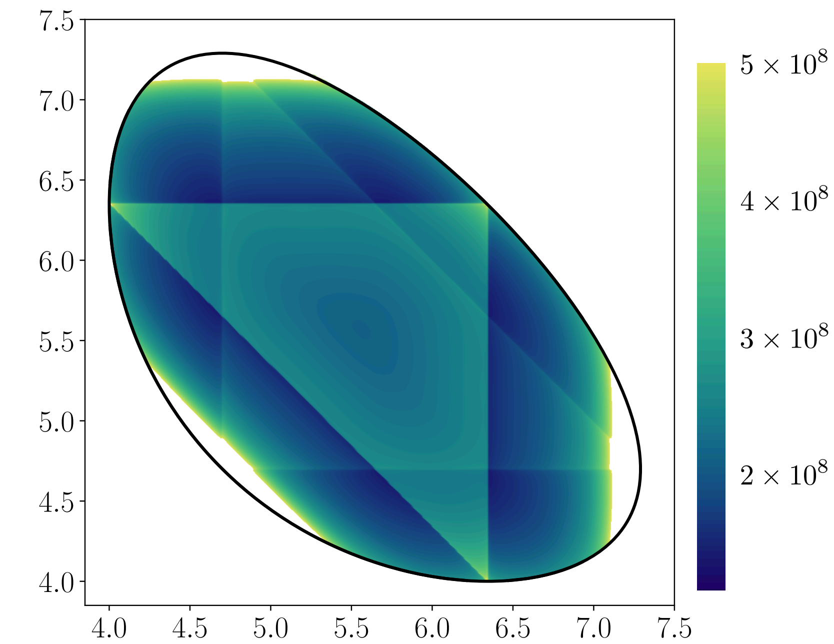
- Generalized Backus-Gilbert takes $\widehat{\delta}_\Delta^{\text{target}}(\bar{\omega},\omega)$ as input
- *Succesful implementation in O(3) model*



Two strategies… Conclusion

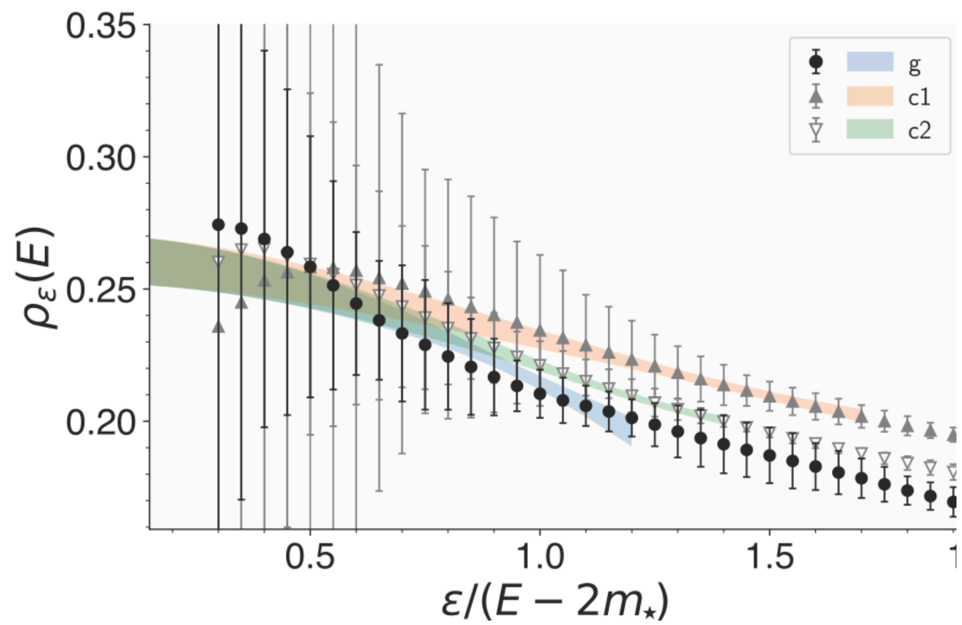
☑ Finite-volume as a tool

- Relate energies and matrix elements
- Tested and highly successful approach
- Limitations:
  modeling/parametrizing in order to fit
  need formalism for all open channels at a given energy



☑ Spectral function method

- More direct/natural in a sense (my opinion)
- No new formalism needed for any energies, channels
- Limitations:
  Tricky volume effects
  Difficult inverse problem



Thanks for listening!… questions?