Three-pion scattering from lattice QCD

Maxwell T. Hansen

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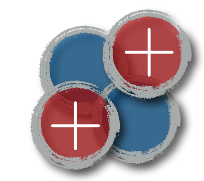
THE UNIVERSITY of EDINBURGH

Recipe for strong force predictions

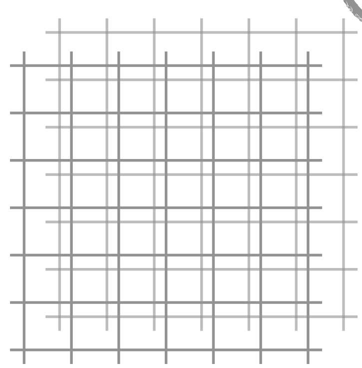
1. Lagrangian defining QCD +
2. Formal / numerical machinery (lattice QCD) +
3. A few experimental inputs (e.g. M_π, M_K, M_Ω) =

$$\mathcal{L}_{\mathrm{QCD}} = \sum_f \overline{\Psi}_f (i\not{D} - m_f)\Psi_f - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a$$



Wide range of precision pre-/post-dictions

$m_q^{\overline{\mathrm{MS}}}(\mu)$ m_p $\alpha_{\overline{\mathrm{MS}}}(\mu)$ m_n f_π g_A f_K $(g-2)_\mu^{\mathrm{HVP}}$ f_D f_B



FLAG 2019 Flavour Lattice Averaging Group

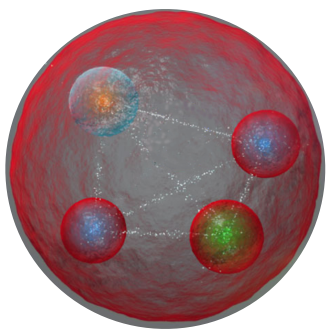
$\alpha^{(5)}_{\overline{\mathrm{MS}}}(M_Z) = 0.1182(8)$ lattice average

$\alpha^{(5)}_{\overline{\mathrm{MS}}}(M_Z) = 0.1174(16)$ PDG 18 (non-lattice)

Overwhelming evidence for QCD ✓ → *Tool for new physics searches*

Multi-hadron observables

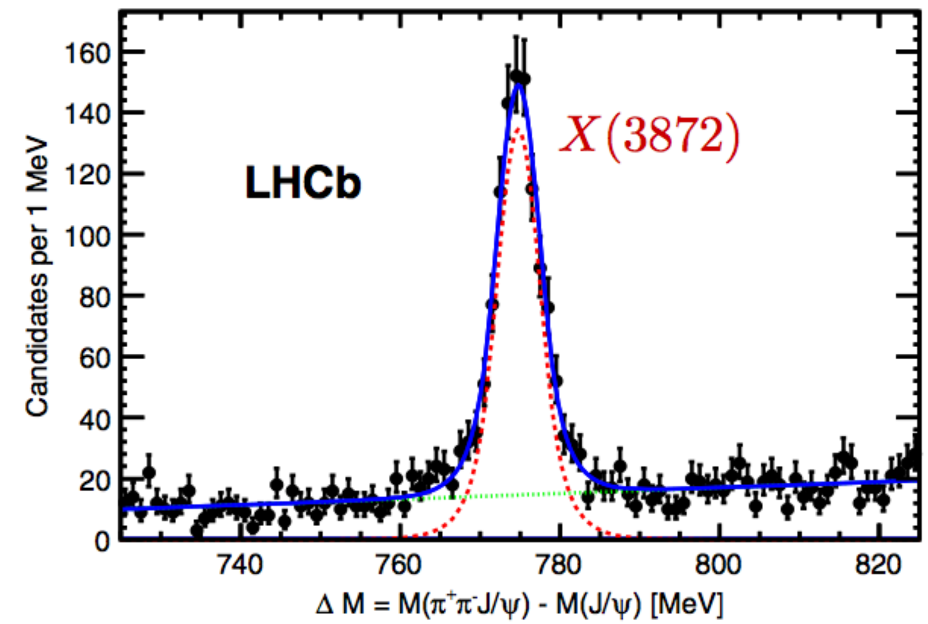
- Exotics, XYZs, tetra- and penta-quarks, *H* dibaryon



e.g. X(3872)

$\sim |D^0\overline{D}^{*0} + \overline{D}^0 D^{*0}\rangle$?

• LHCb (PRD92, 2015) •



- Eletroweak, CP violation, resonant enhancement

CP violation in charm $D \to \pi\pi, K\overline{K}$

$$\Delta A_{CP} = (-15.4 \pm 2.9) \times 10^{-4}$$

• LHCb (PRL, 2019) •

$f_0(1710)$ could enhance ΔA_{CP}

• Soni (2017) •

Resonant B decays $B \to K^* \ell\ell \to K\pi\,\ell\ell$

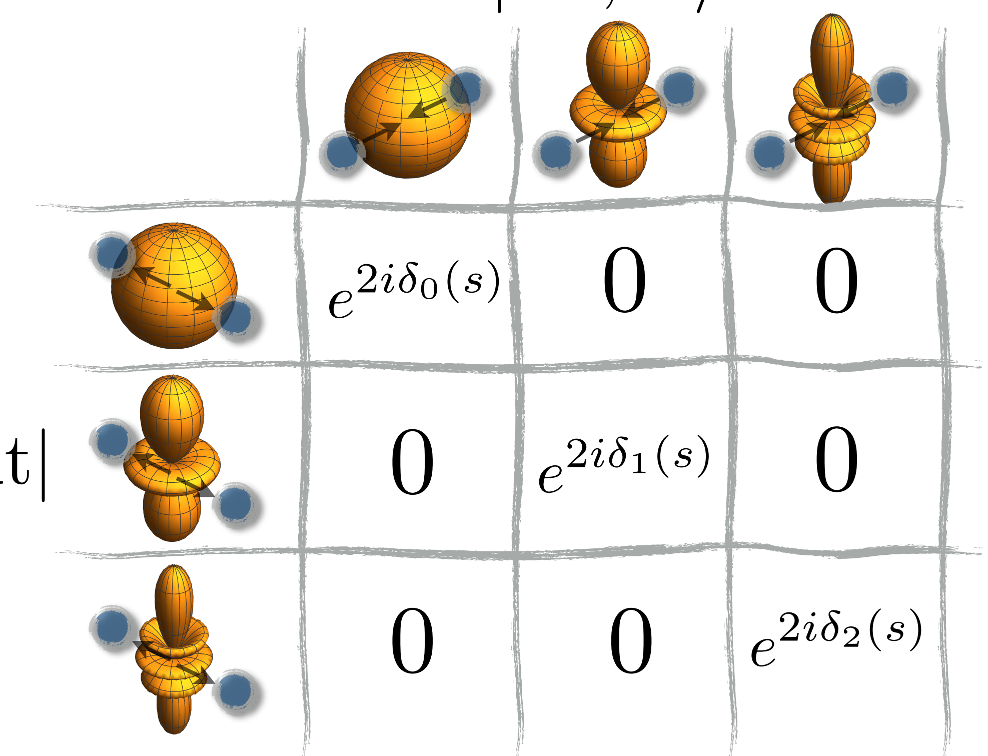
$|X\rangle, |\rho\rangle, |K^*\rangle, |f_0\rangle \quad \notin$ **QCD Fock space**

QCD Fock space

- At low-energies QCD = hadronic degrees of freedom $\pi \sim \bar{u}d,\ K \sim \bar{s}u,\ p \sim uud$
- Overlaps of multi-hadron ***asymptotic states*** → S matrix

$|\pi\pi, \mathrm{in}\rangle$

$S(s) \equiv \langle \pi\pi, \mathrm{out}|$



depends on $s = E_{\mathrm{cm}}^2$
and angular variables

diagonal in angular momentum

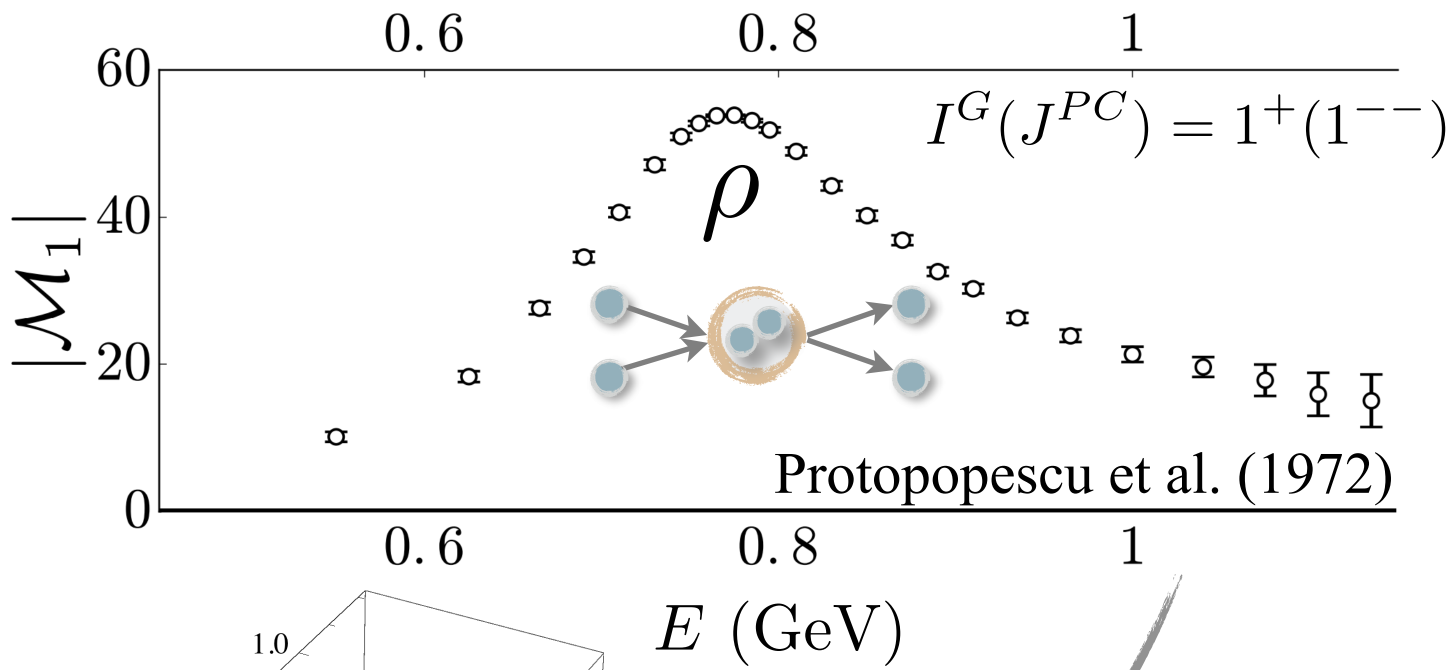
$$\mathcal{M}_\ell(s) \propto e^{2i\delta_\ell(s)} - 1$$

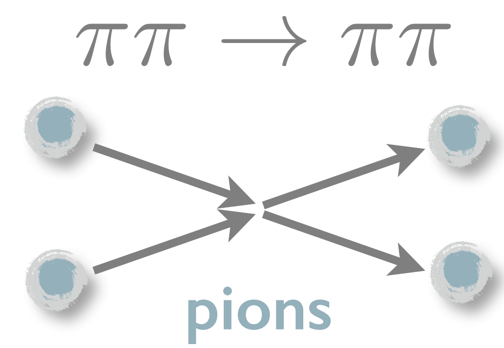
- ***An enormous space of information*** $|\pi\pi\pi\pi, \mathrm{in}\rangle \quad |K\overline{K}, \mathrm{in}\rangle \quad \cdots$

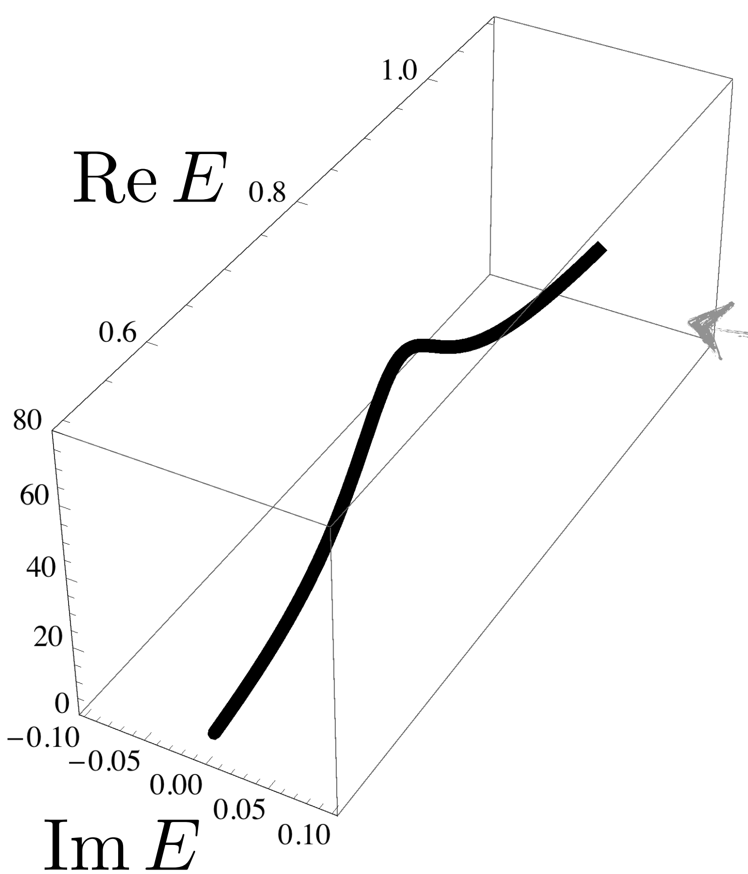
QCD resonances

- Roughly speaking, a bump in: $|\mathcal{M}_\ell(s)|^2 \propto |e^{2i\delta_\ell(s)} - 1|^2 \propto \sin^2 \delta_\ell(s)$

scattering rate





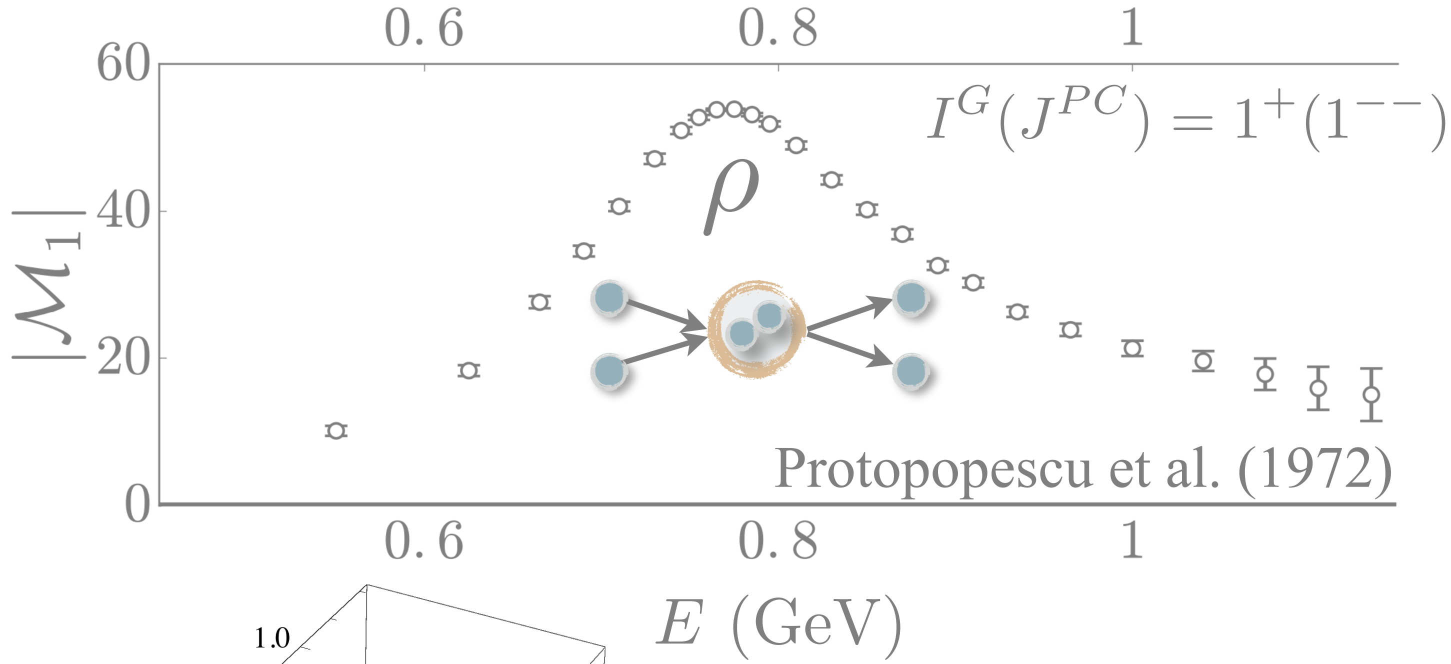


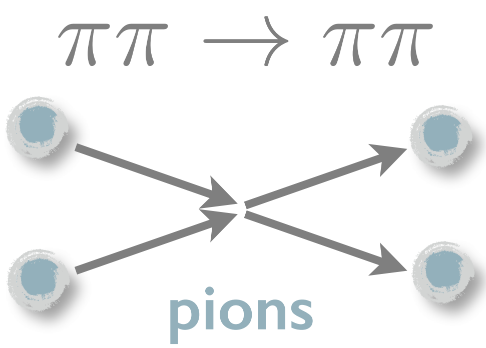
Extend to the **complex plane**

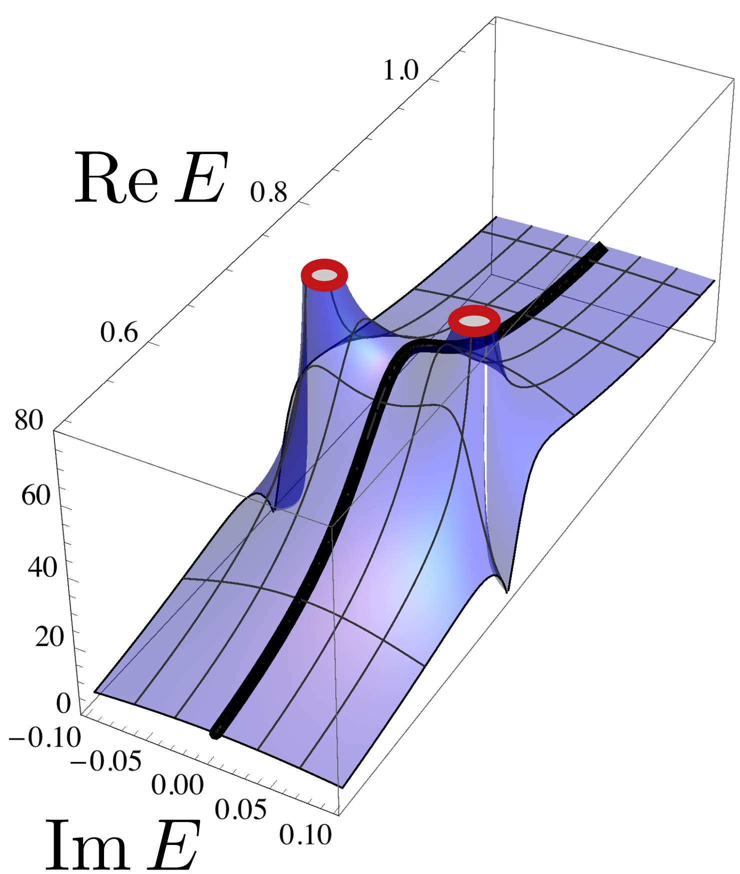
QCD resonances

- ☐ Roughly speaking, a bump in: $|\mathcal{M}_\ell(s)|^2 \propto |e^{2i\delta_\ell(s)} - 1|^2 \propto \sin^2 \delta_\ell(s)$

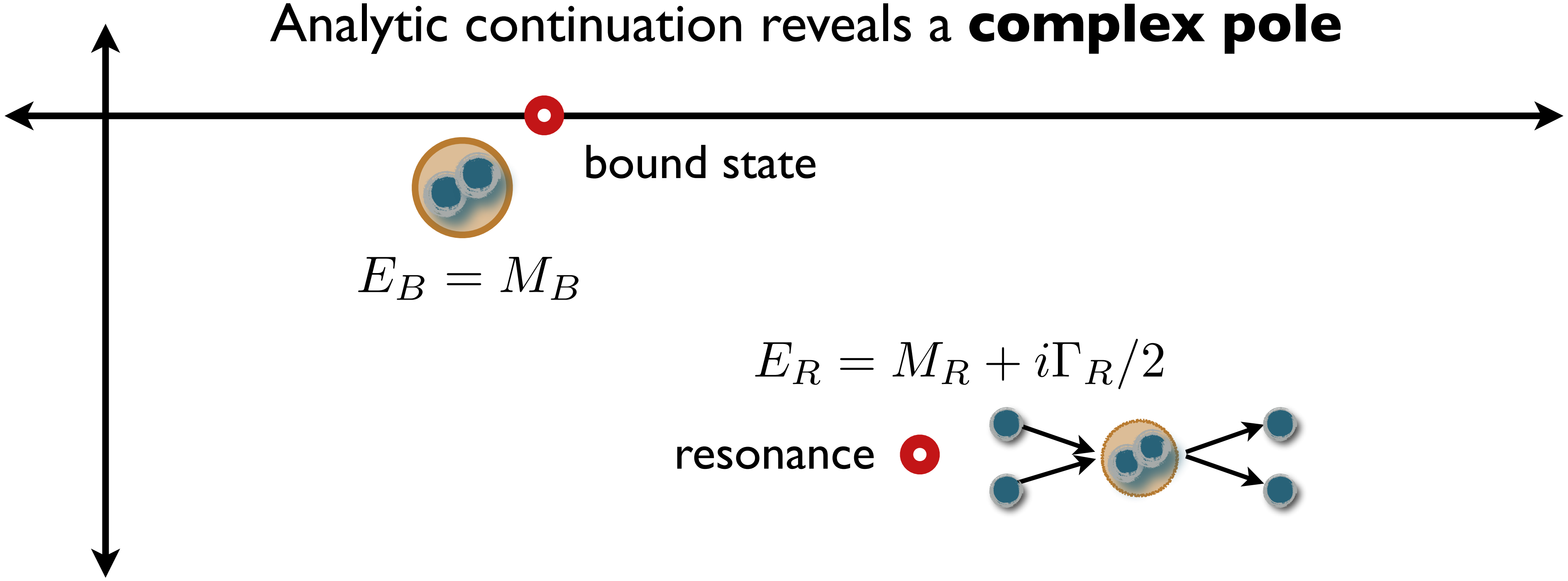
scattering rate







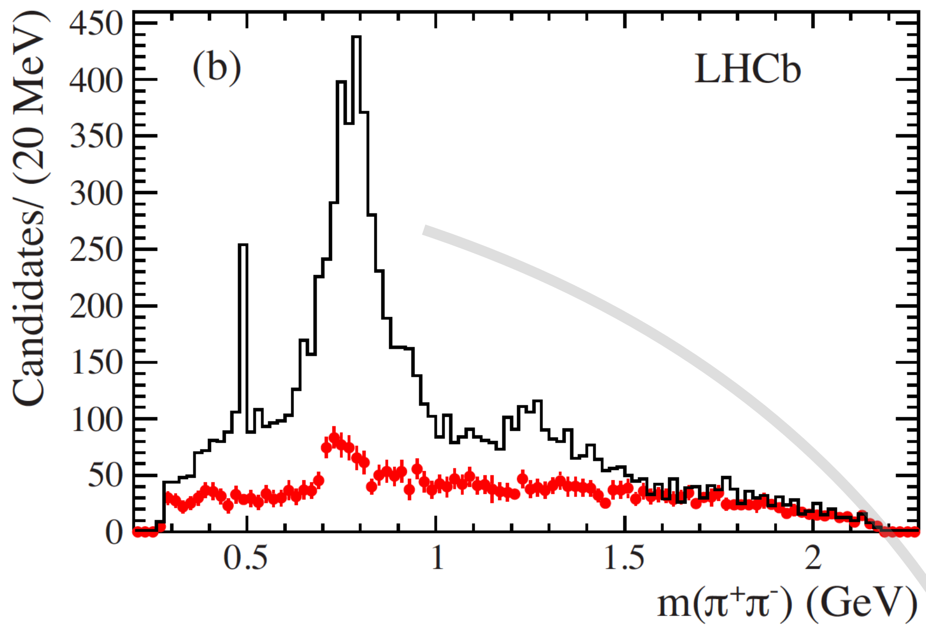
Analytic continuation reveals a **complex pole**



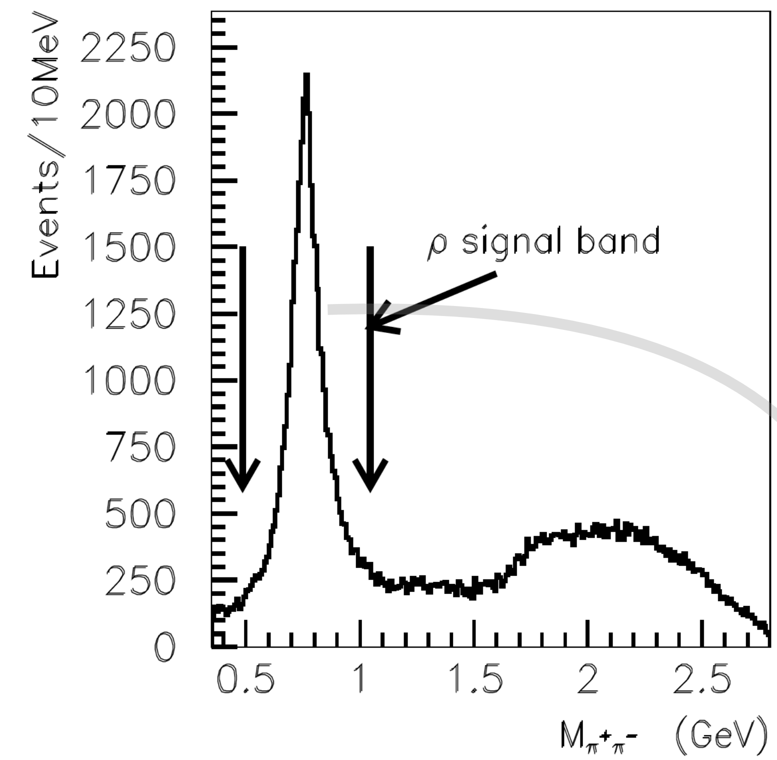
Pole is universal

- Resonances often seen in "production"

$\overline{B}^0 \to J/\psi\,\pi^+\pi^-$

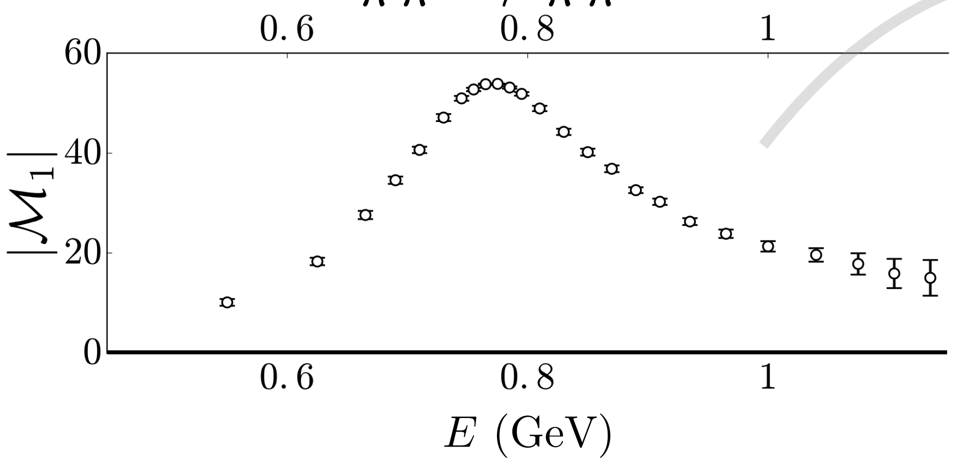


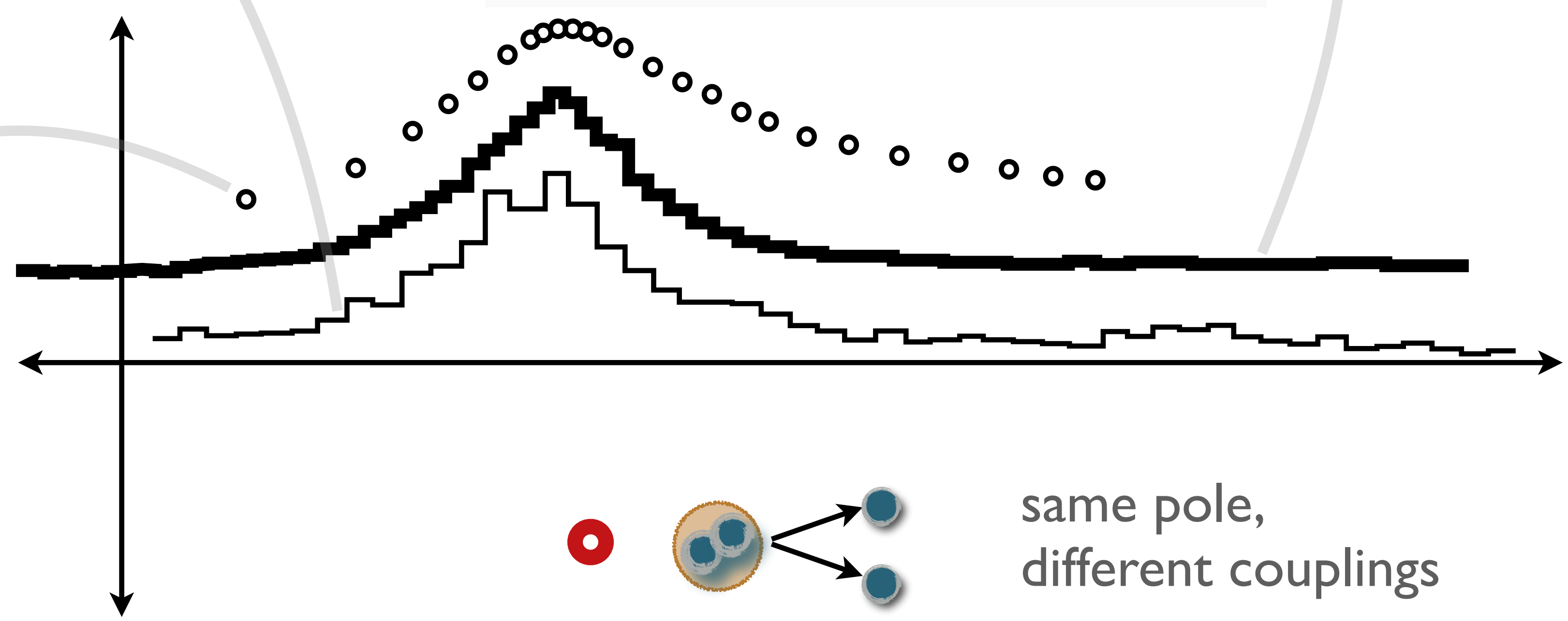
$J/\psi \to \gamma\gamma\rho$



(as opposed to scattering)

$\pi\pi \longrightarrow \pi\pi$

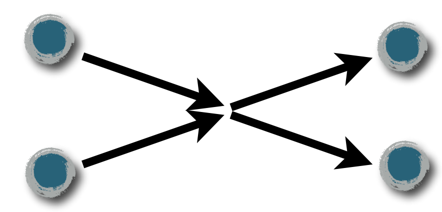




Analyticity

- Instead of $|\mathcal{M}(s)|^2$ → analytically continue the ***amplitude*** itself

  For two-particle energies $(2m)^2 < s < (4m)^2$, what is the analytic structure?



- The optical theorem tells us…

$$\rho(s)|\mathcal{M}_\ell(s)|^2 = \mathrm{Im}\,\mathcal{M}_\ell(s)$$

where $\rho(s) = \dfrac{\sqrt{1-4m^2/s}}{32\pi}$ is the two-particle phase space

- Unique solution is…

$$\mathcal{M}_\ell(s) = \frac{1}{\mathcal{K}_\ell(s)^{-1} - i\rho(s)}$$

K matrix (short distance)

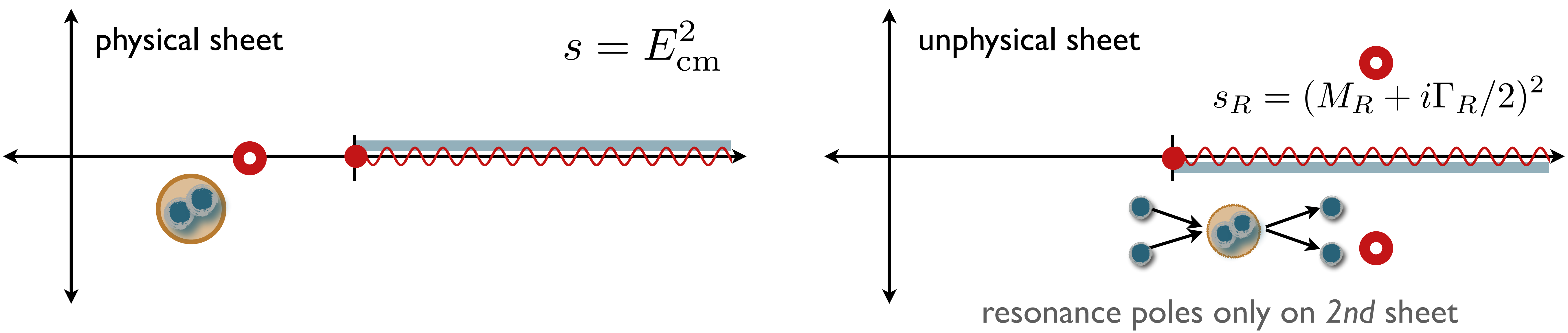
phase-space cut (long distance)

Key message: *The scattering amplitude has a square-root branch cut*

Cuts and sheets

$$\mathcal{M}_\ell(s) = \frac{1}{\mathcal{K}_\ell(s)^{-1} - \rho(s)} \propto \frac{1}{p\cot\delta_\ell(s) - ip} \propto e^{2i\delta_\ell(s)} - 1 \qquad \rho(s) \propto i\sqrt{s-(2m)^2}$$

- Each channel generates a *square-root cut* → doubles the number of sheets



- Important lessons:

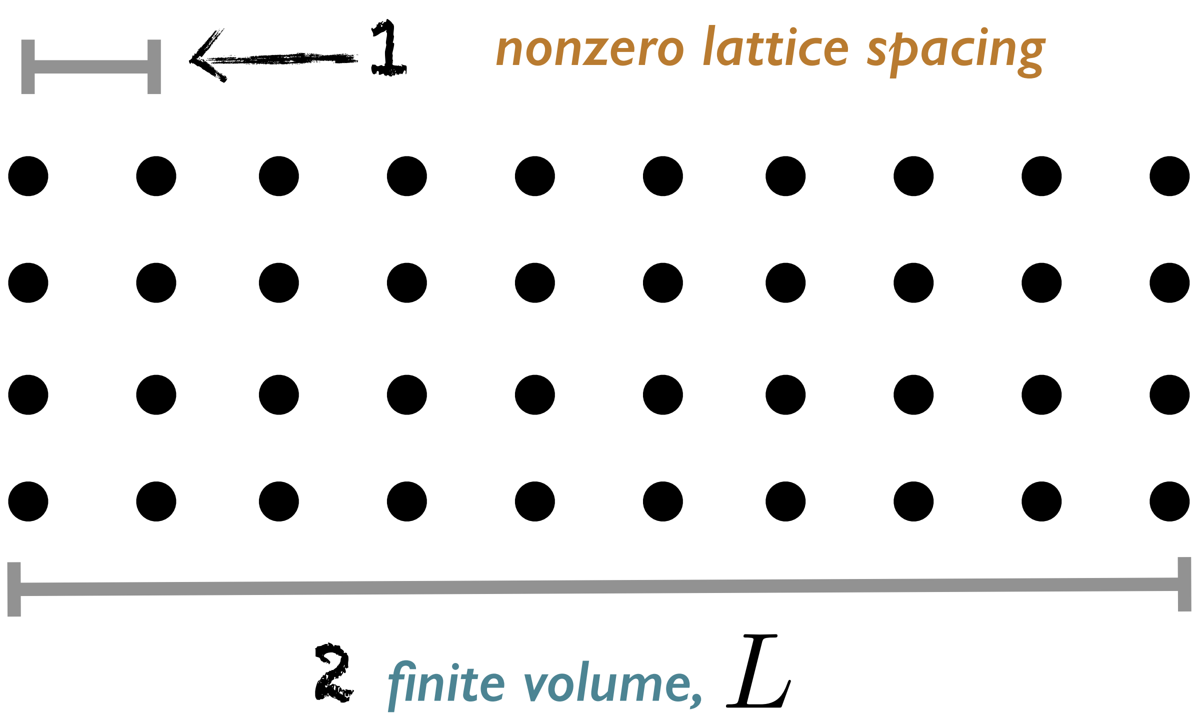
Details of analyticity = important for quantitative understanding

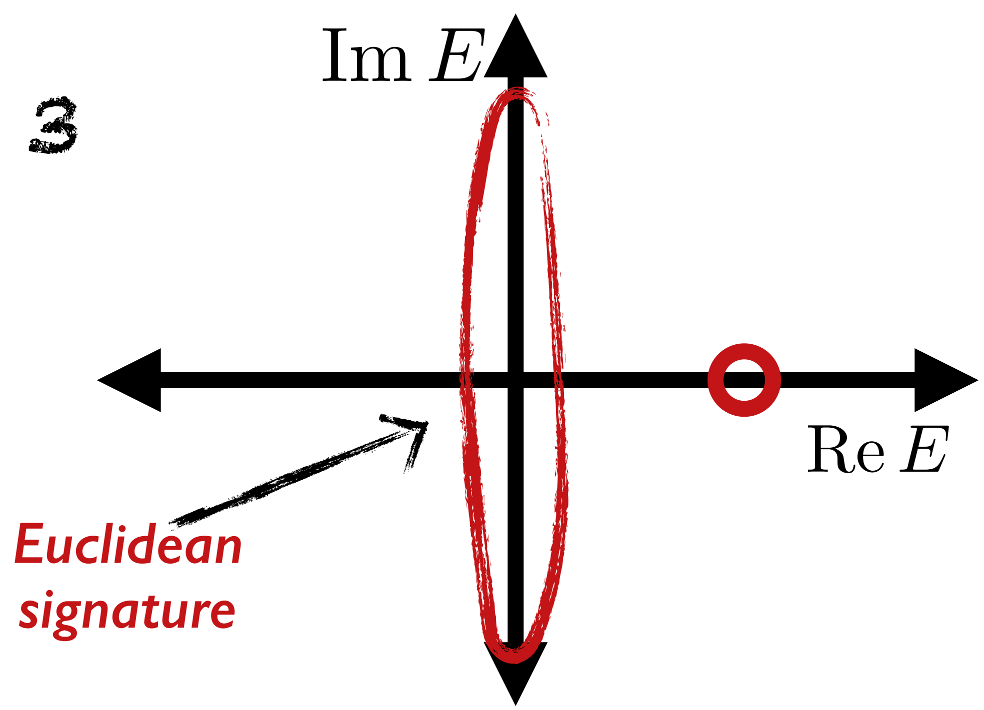
Possible to separate…
(i) long-distance kinematic singularities
(ii) short-distance/microscopic physics (depending on interaction details)

Microscopic physics *via Lattice QCD*

$$\text{observable?} = \int d^N\phi \, e^{-S} \left[\begin{array}{c} \textbf{interpolator} \\ \textbf{for observable} \end{array} \right]$$

To proceed we have to make *three modifications*

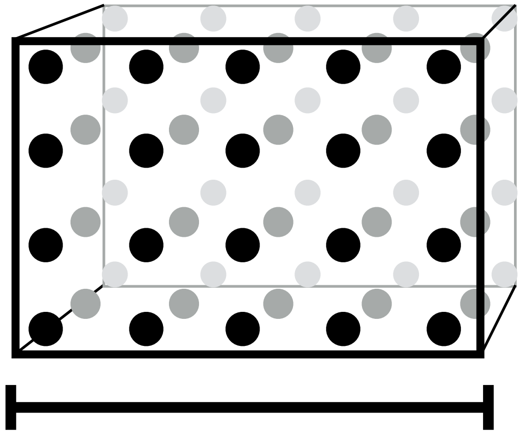




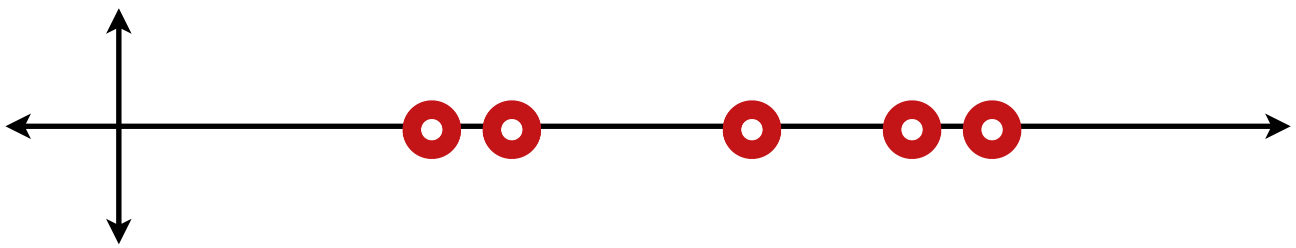
Also… $M_{\pi,\text{lattice}} > M_{\pi,\text{our universe}}$

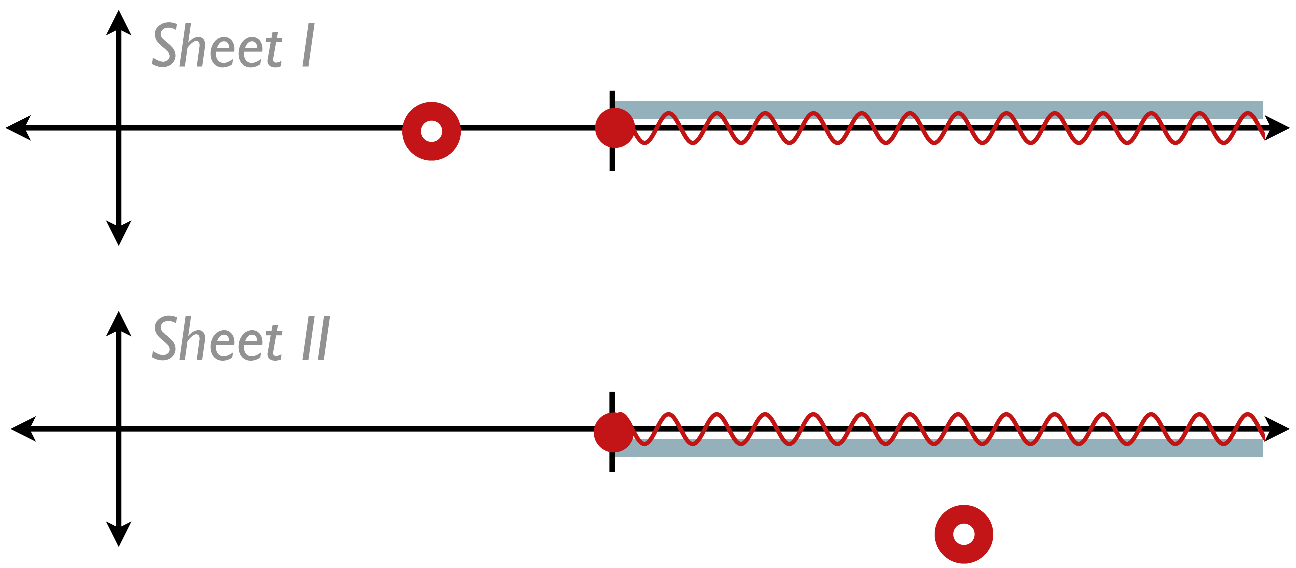
(but physical masses → increasingly common)

Difficulties for multi-hadron observables



- The ***finite volume***...
  - *Discretizes* the spectrum
  - *Eliminates* the branch cuts and extra sheets
  - *Hides* the resonance poles





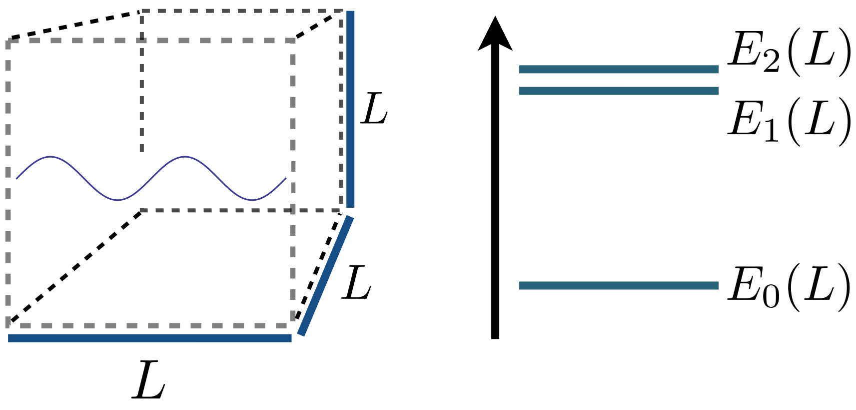
- LQCD → Energies and matrix elements

$$\langle \mathcal{O}_j(\tau)\mathcal{O}_i^\dagger(0)\rangle = \sum_n \langle 0|\mathcal{O}_j(\tau)|E_n\rangle\langle E_n|\mathcal{O}_i^\dagger(0)|0\rangle = \sum_n e^{-E_n(L)\tau} Z_{n,j} Z_{n,i}^*$$

- Our task is relate $E_n(L)$ and $\langle E_{m'}|\mathcal{J}(0)|E_m\rangle$ to ***experimental observables***

The finite-volume as a tool

- Finite-volume set-up

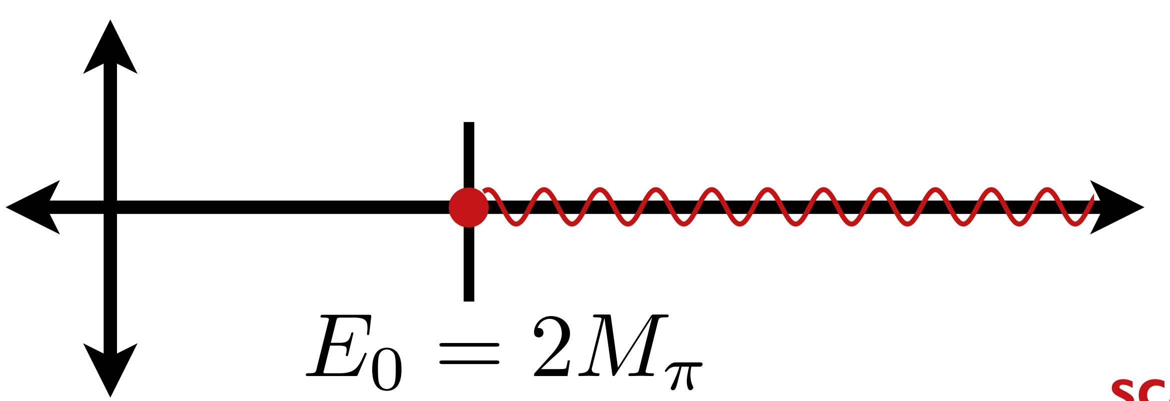


- **cubic**, spatial volume (extent L)
- **periodic**

$$\vec{p} = \frac{2\pi}{L}\vec{n}\,, \qquad \vec{n} \in \mathbb{Z}^3$$

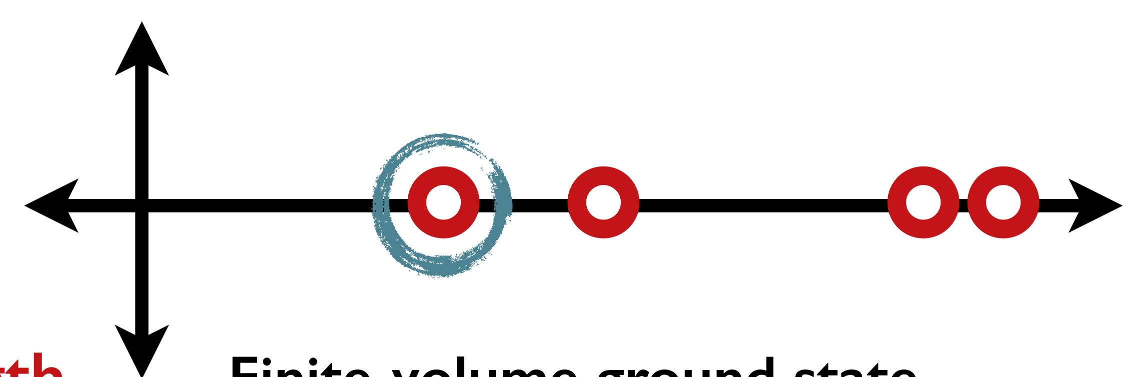
- L is large enough to neglect $e^{-M_\pi L}$
- T and lattice also negligible

- Scattering leaves an *imprint* on finite-volume quantities



$E_0 = 2M_\pi$

Infinite-volume threshold



Finite-volume ground state

scattering length

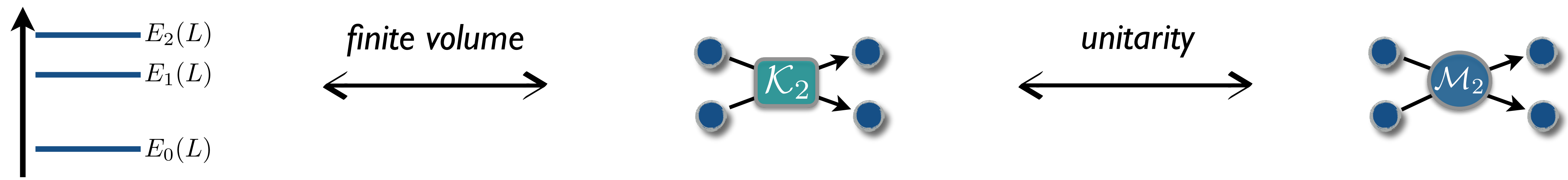
$$\mathcal{M}_{\ell=0}(2M_\pi) = -32\pi M_\pi a$$

$$E_0(L) = 2M_\pi + \frac{4\pi a}{M_\pi L^3} + \mathcal{O}(1/L^4)$$

• Huang, Yang (1958) •

General relation

$$\det[\mathcal{K}^{-1}(s) + F(P, L)] = 0 \qquad F(P,L) \equiv \text{Matrix of known geometric functions}$$



Holds only for two-particle energies $s < (4m)^2$ Neglects e^{-mL}

Generalized to *non-degenerate masses, multiple channels, spinning particles*

Encodes angular momentum mixing

• Lüscher (1989) • *many others* •

Scattering processes and resonances from lattice QCD

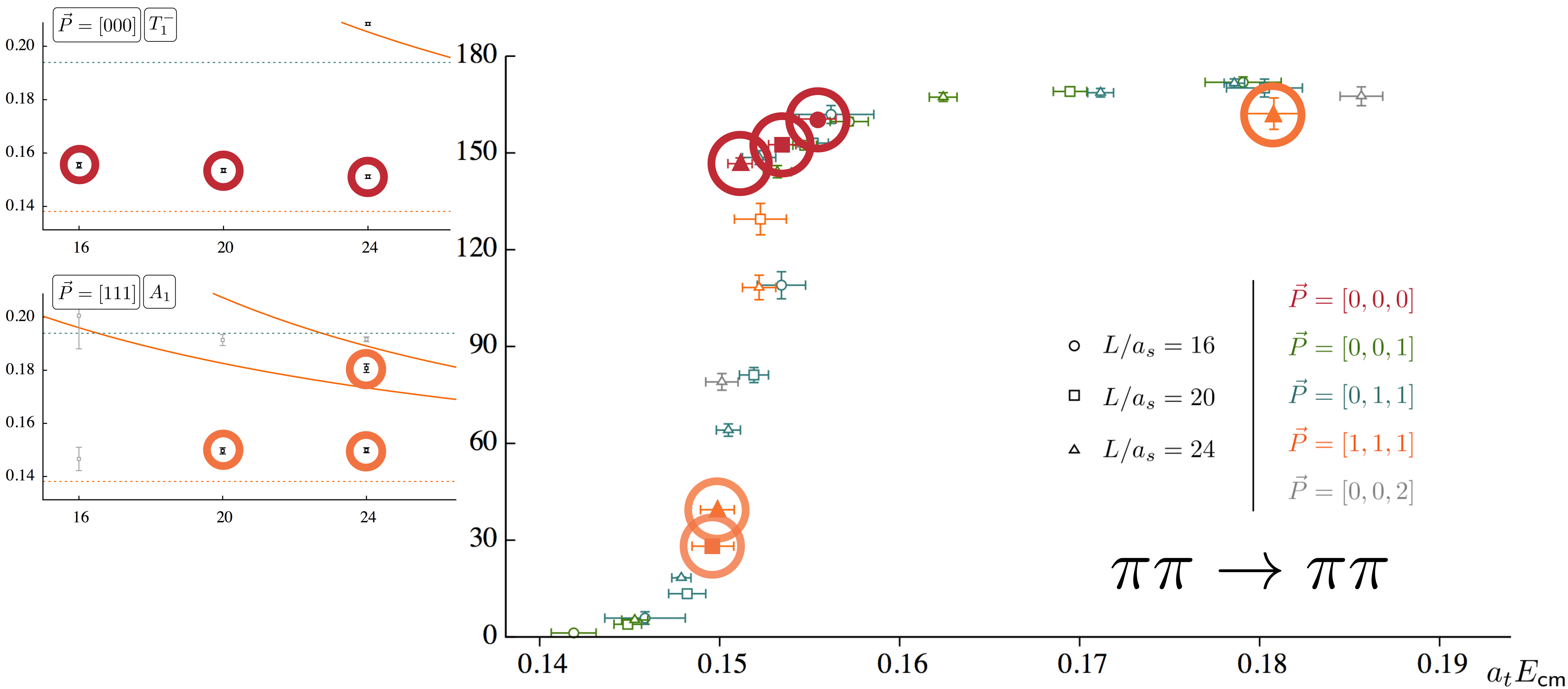
Raúl A. Briceño,[1,*] Jozef J. Dudek,[1,2,†] and Ross D. Young[3,‡]

REVIEWS OF MODERN PHYSICS

Using the result

- Single-channel case *(pions in a p-wave)*

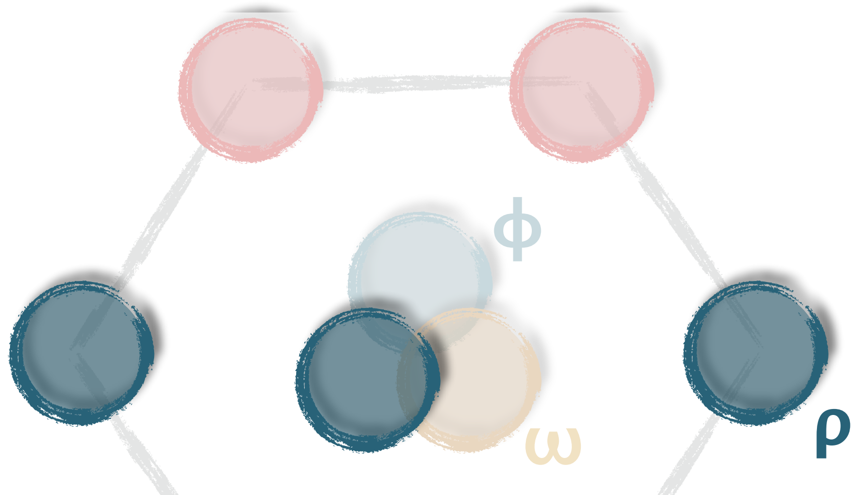
$$\mathcal{K}(s_n)^{-1} = \rho \cot \delta(s_n) = -F(E_n, \vec{P}, L)$$

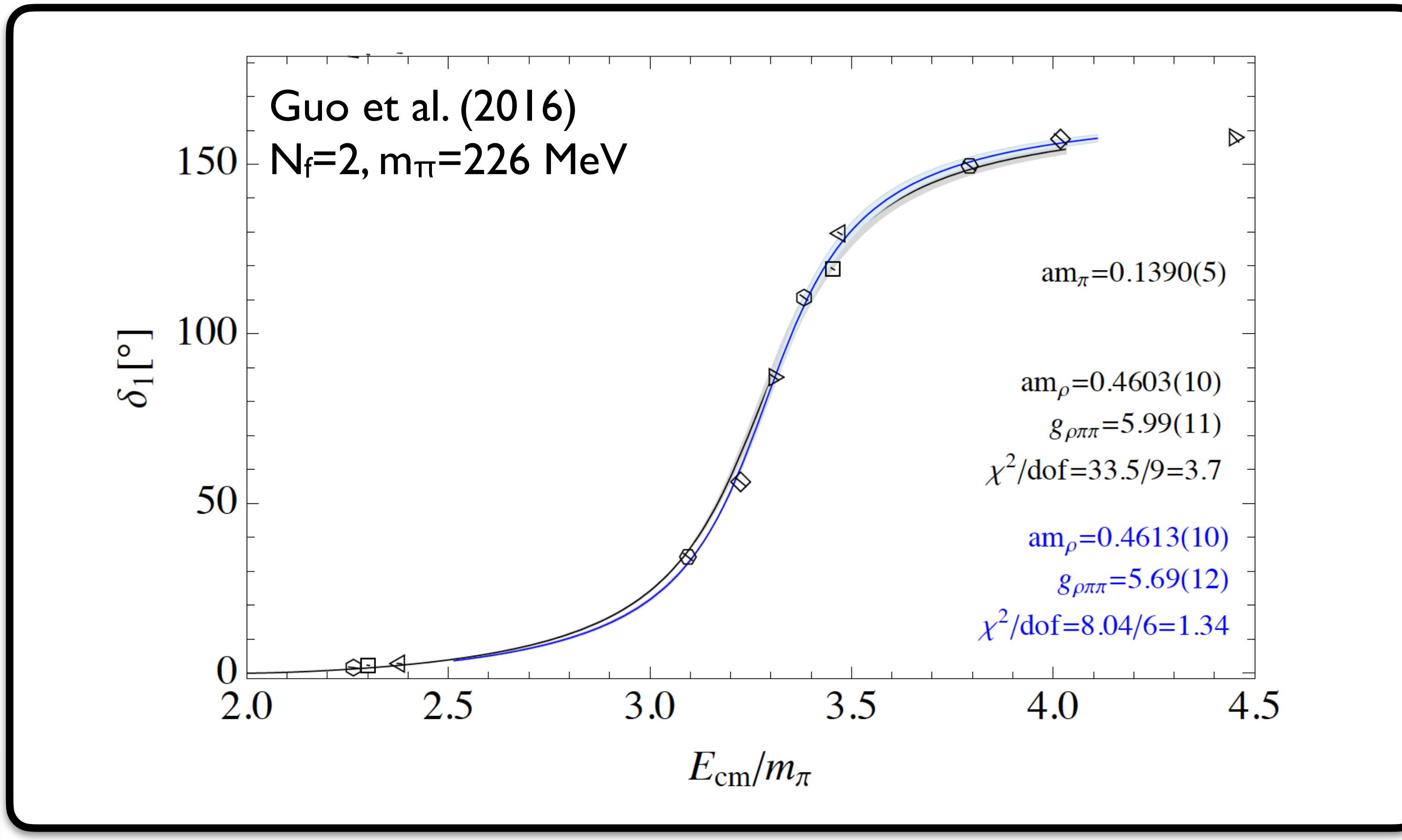


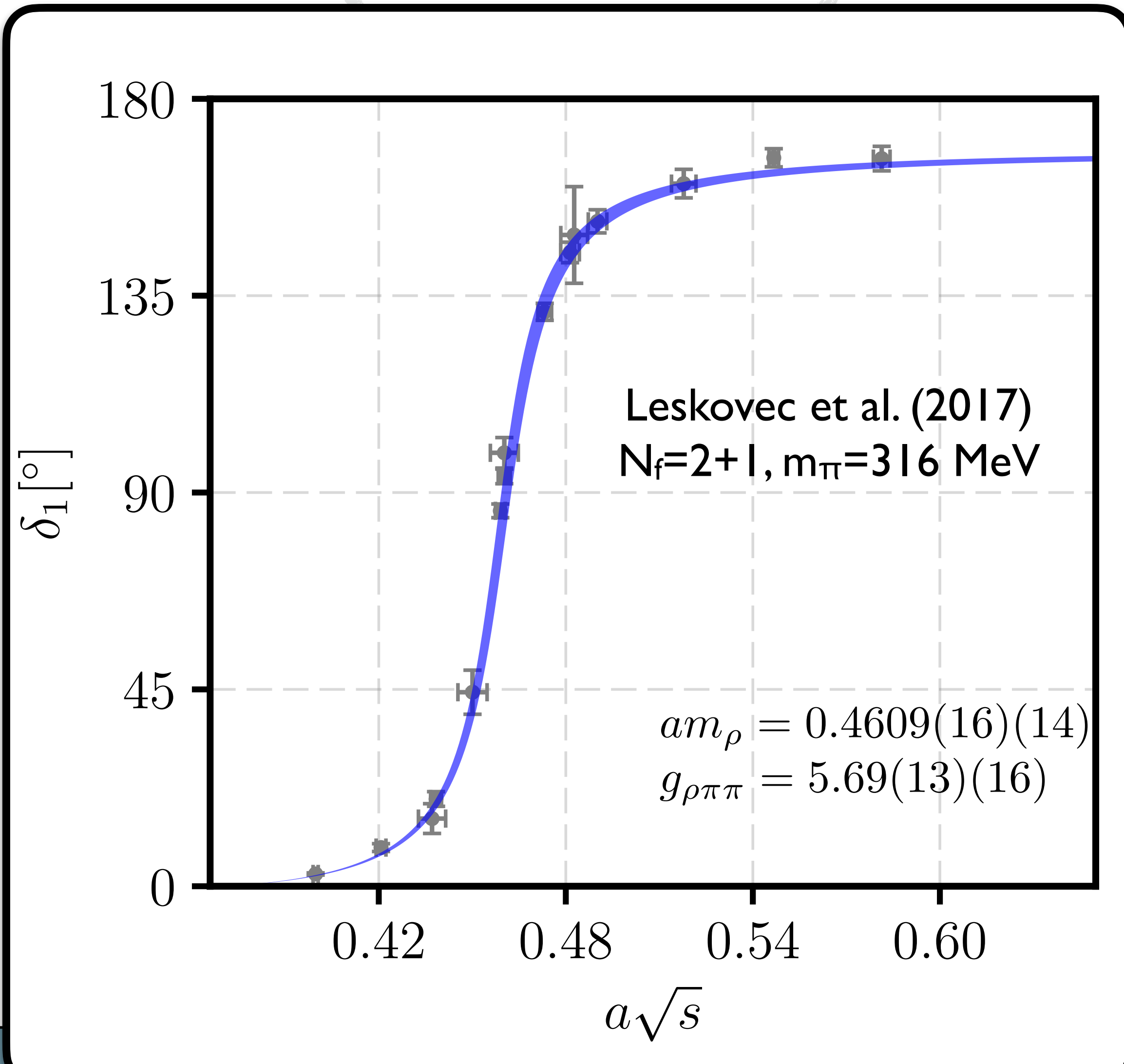
• Dudek, Edwards, Thomas in *Phys.Rev.* D87 (2013) 034505 •

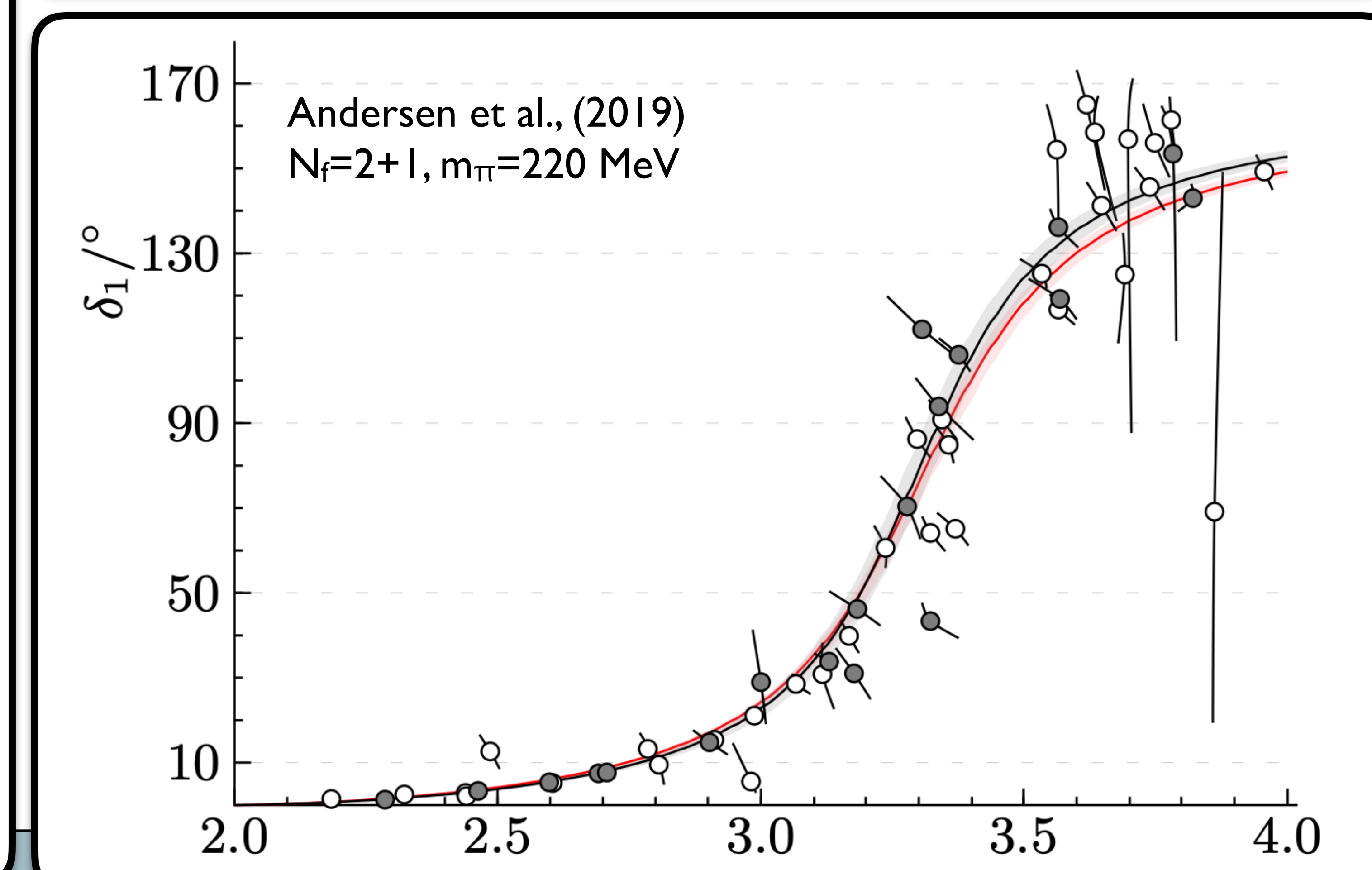
$$\rho \to \pi\pi$$

$$I^G(J^{PC}) = 1^+(1^{--})$$





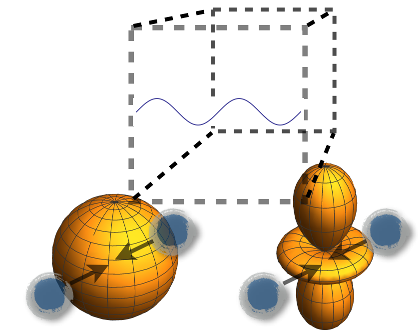




Coupled channels

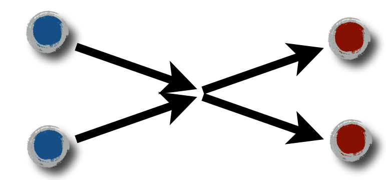
- The cubic volume mixes different partial waves…

e.g. $K\pi \to K\pi$, $\vec{P} \neq 0$ $\longrightarrow$ $$\det\left[\begin{pmatrix}\mathcal{K}_s^{-1} & 0 \\ 0 & \mathcal{K}_p^{-1}\end{pmatrix} + \begin{pmatrix}F_{ss} & F_{sp} \\ F_{ps} & F_{pp}\end{pmatrix}\right] = 0$$



…as well as different flavor channels…

e.g. $a = \pi\pi$, $b = K\overline{K}$ $\longrightarrow$ $$\det\left[\begin{pmatrix}\mathcal{K}_{a\to a} & \mathcal{K}_{a\to b} \\ \mathcal{K}_{b\to a} & \mathcal{K}_{b\to b}\end{pmatrix}^{-1} + \begin{pmatrix}F_a & 0 \\ 0 & F_b\end{pmatrix}\right] = 0$$

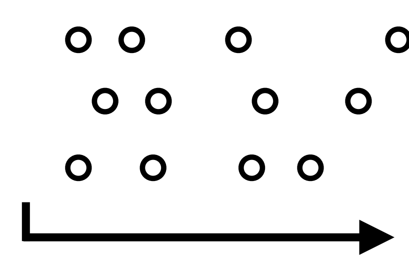


- Workflow…

Correlators with a large operator basis
$\langle \mathcal{O}_a(\tau)\mathcal{O}_b^\dagger(0)\rangle$

→ Reliably extract finite-volume energies
$\langle \Omega_m(\tau)\Omega_m^\dagger(0)\rangle \sim e^{-E_m(L)\tau}$

→ Vary L and P to recover a dense set of energies



↓

hadspec

Identify a broad list of K-matrix parametrizations
polynomials and poles | EFT based | dispersion theory based

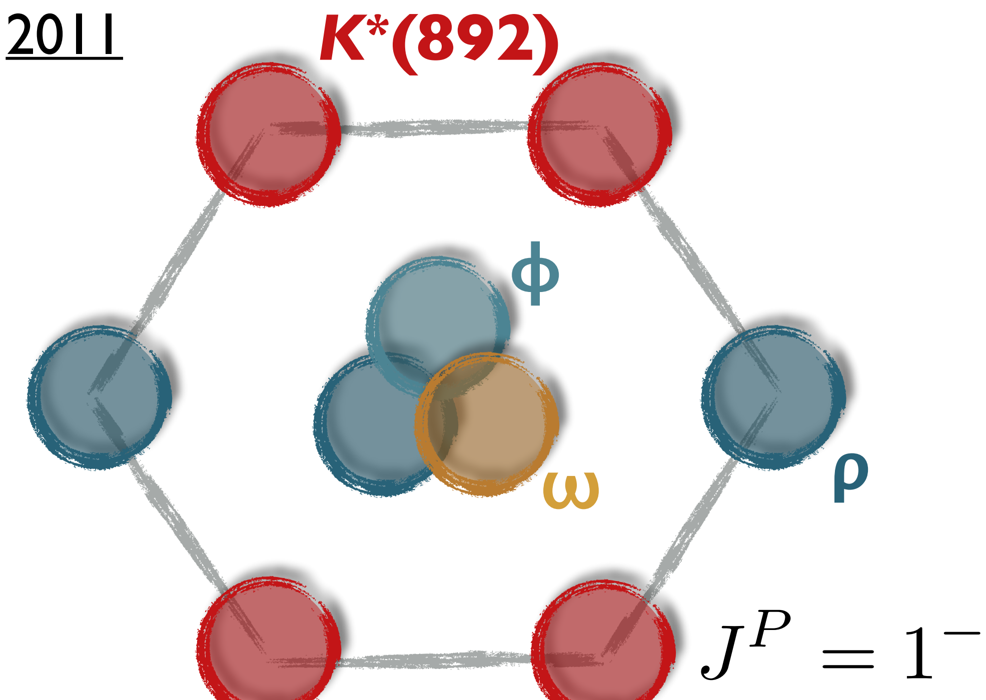
→ **Perform global fits to the finite-volume spectrum**

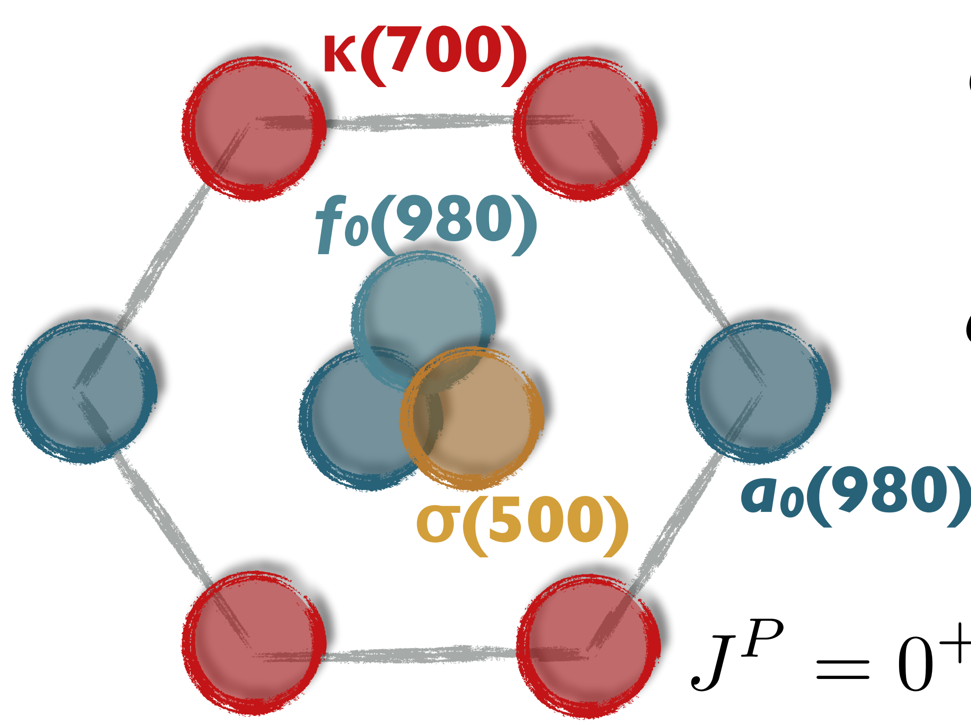
$\rho \to \pi\pi$

- CP-PACS/PACS-CS 2007, 2011
- ETMC 2010
- Lang et al. 2011
- HadSpec 2012, **2016**
- Pellisier 2012
- RQCD 2015
- Guo et al. 2016
- Fu et al. 2016
- Bulava et al. 2016
- Alexandrou et al. 2017
- Andersen et al. 2018
- Fischer et al. 2020
- Erben et al. 2020

$\sigma \to \pi\pi$

- Prelovsek et al. 2010
- Fu 2013
- Wakayama 2015
- Howarth and Giedt 2017
- Briceño et al. 2017
- Guo et al. 2018





$\kappa \to K\pi$

$K^* \to K\pi$

- Lang et al. 2012
- Prelovsek et al. 2013
- **Wilson et al. 2015**
- RQCD 2015
- **Brett et al. 2018**
- Wilson et al. 2019
- Rendon et al. 2020

$b_1 \to \pi\omega, \pi\phi$

- **Woss et al. 2019**

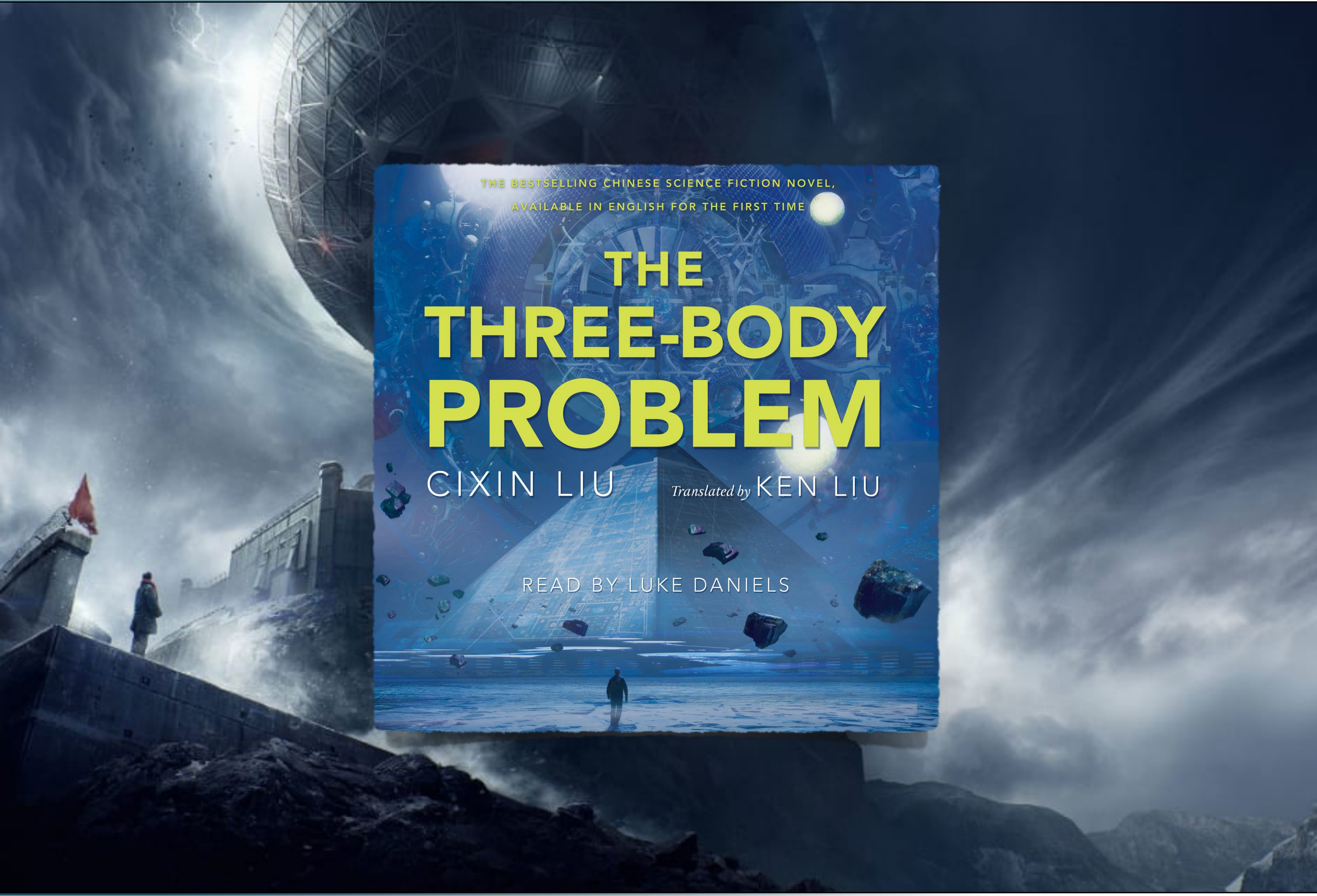
$a_0(980) \to \pi\eta, K\overline{K}$

- **Dudek et al. 2016**

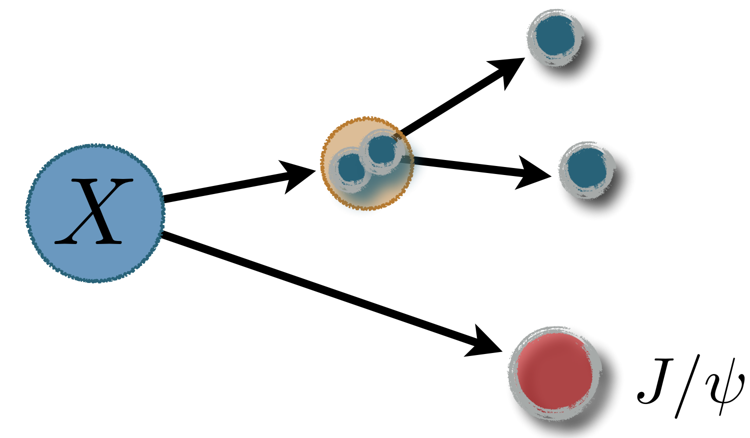
$\sigma, f_0, f_2 \to \pi\pi, K\overline{K}, \eta\eta$

- **Briceño et al. 2017**

See the recent review by Briceño, Dudek and Young


THE BESTSELLING CHINESE SCIENCE FICTION NOVEL,
AVAILABLE IN ENGLISH FOR THE FIRST TIME
THE THREE-BODY PROBLEM
CIXIN LIU
Translated by KEN LIU
READ BY LUKE DANIELS

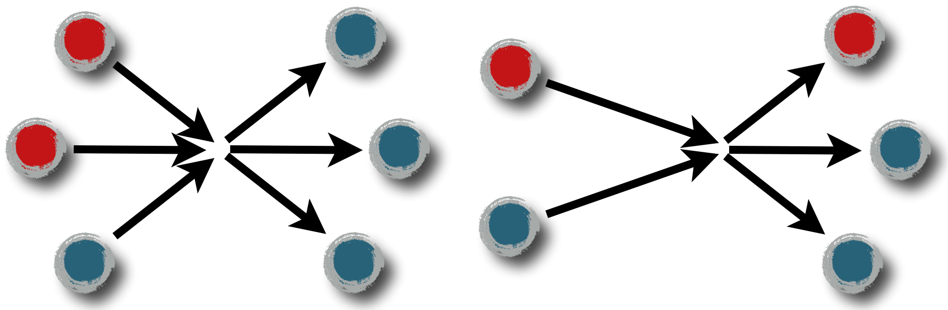
3-particle amplitudes



2-to-2 only samples $J^P \quad 0^+ \quad 1^- \quad 2^+ \cdots$

many interesting resonances have significant 3-body decays

Goal: ***finite-volume + unitarity formalism*** for generic two- and three-particle systems



Applications…

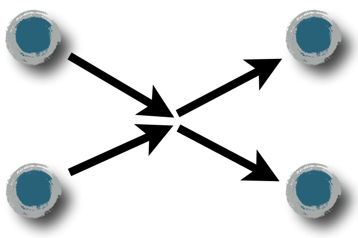
exotic resonance pole positions, couplings, quantum numbers

$\omega(782),\ a_1(1420) \to \pi\pi\pi \qquad X(3872) \to J/\psi\pi\pi \qquad X(3915)[Y(3940)] \to J/\psi\pi\pi$

form factors and transitions

and much more!… *(3-body forces, weak transitions, gluons content)*

Complication: degrees of freedom

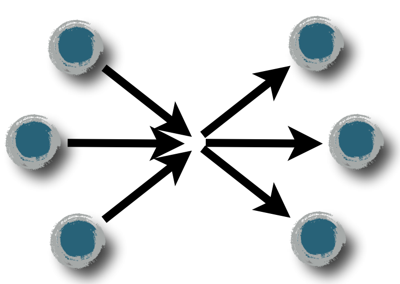


12 momentum components

-10 Poincaré generators

2 degrees of freedom

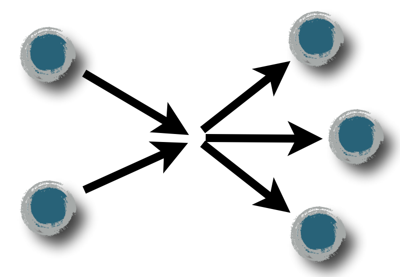
$\vec{p}_1 + \vec{p}_2 \;\rightarrow\; \vec{p}_3 + \vec{p}_4 \;\longrightarrow$ Mandelstam s, t



18 momentum components

-10 Poincaré generators

8 degrees of freedom



15 momentum components

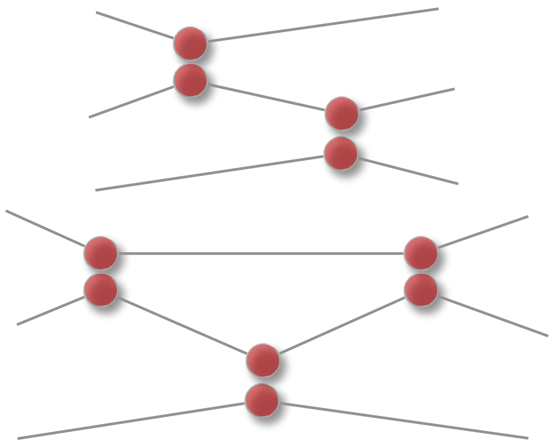
-10 Poincaré generators

5 degrees of freedom

Complication: on-shell states

- Classical pairwise scattering

for $m_1 = m_2 = m_3$ up to ***3*** binary collisions are possible



Dispersion Relations for Three-Particle Scattering Amplitudes. I*

MORTON RUBIN
Physics Department, University of Wisconsin, Madison, Wisconsin

AND

ROBERT SUGAR
Physics Department, Columbia University, New York, New York

AND

GEORGE TIKTOPOULOS
Palmer Physical Laboratory, Princeton University, Princeton, New Jersey
(Received 31 January 1966)

$$b = \frac{(m_1+m_3)(m_2+m_3)}{m_1 m_2}$$

It follows that if

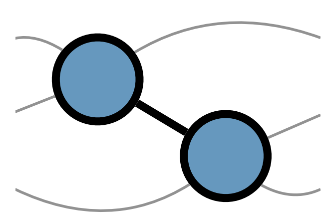
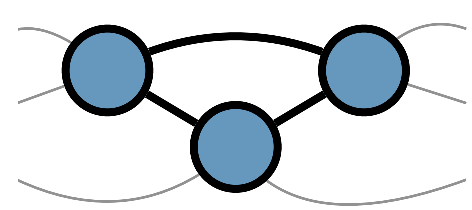
$$b^{n-2}(b-1) > 1\,, \qquad \text{(IV.18)}$$

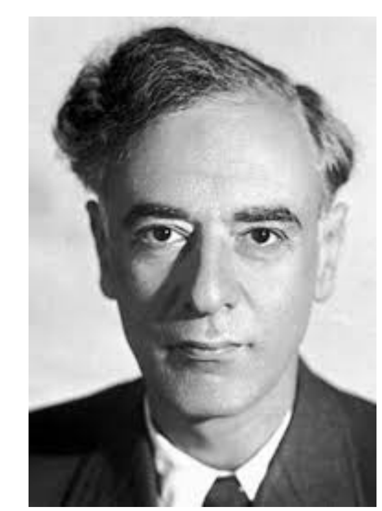
then $2n+1$ successive binary collisions are kinematically impossible.

$m_1 = m_2 = m_3$ - ε:
4 collisions possible
$\pi\pi K$

b < 2
5 collisions possible
$\pi K K$

- Correspond to Landau singularities

$i\mathcal{M}_{3\to 3} \equiv$ fully connected correlator $=$  $+$  $+ \cdots$



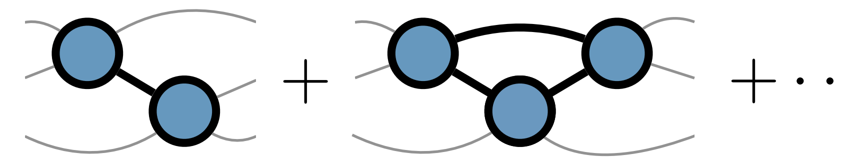
complicate analyticity & unitarity

difficult to disentangle kinematic singularities from resonance poles

Two key observations

- Intermediate $K_{\mathrm{df},3}$ removes singularities

$\mathcal{K}_{\mathrm{df},3} \equiv$ fully connected diagrams w/ ***PV pole prescription*** $-$



same degrees of freedom as *M3*

smooth real function

relation to *M3* = known

- $K_{\mathrm{df},3}$ has a systematic low-energy expansion

$$\mathcal{K}_{\mathrm{df},3}(p_3, p_2, p_1; k_3, k_2, k_1) = \mathcal{K}_{\mathrm{df},3}^{\mathrm{iso},0} + \mathcal{K}_{\mathrm{df},3}^{\mathrm{iso},1}\,\Delta + \cdots \qquad \Delta = \frac{s-(3m)^2}{(3m)^2}$$

smooth real function

analogous to effective range expansion

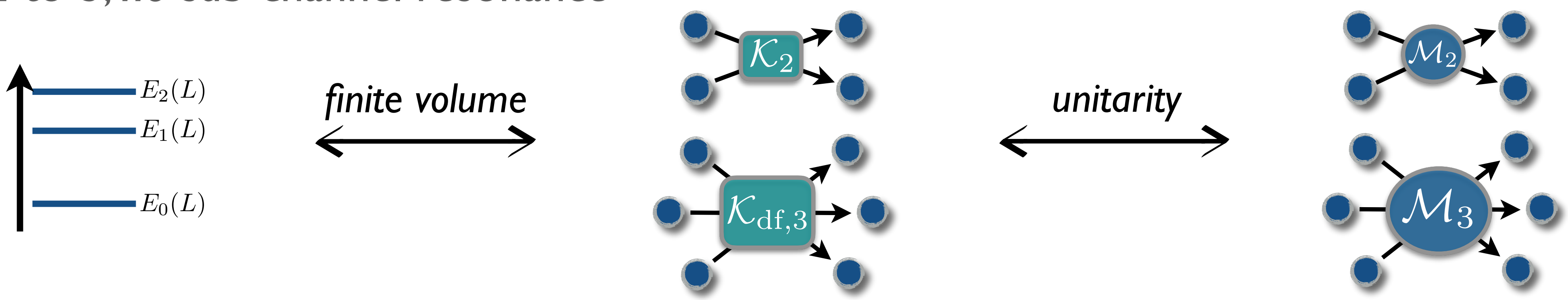
$$p\cot\delta = -\frac{1}{a} + \frac{1}{2}\, r\, p^2 + \mathcal{O}(p^4)$$

gives handle on many *degrees of freedom*
(DOFs enter order by order)

Status…

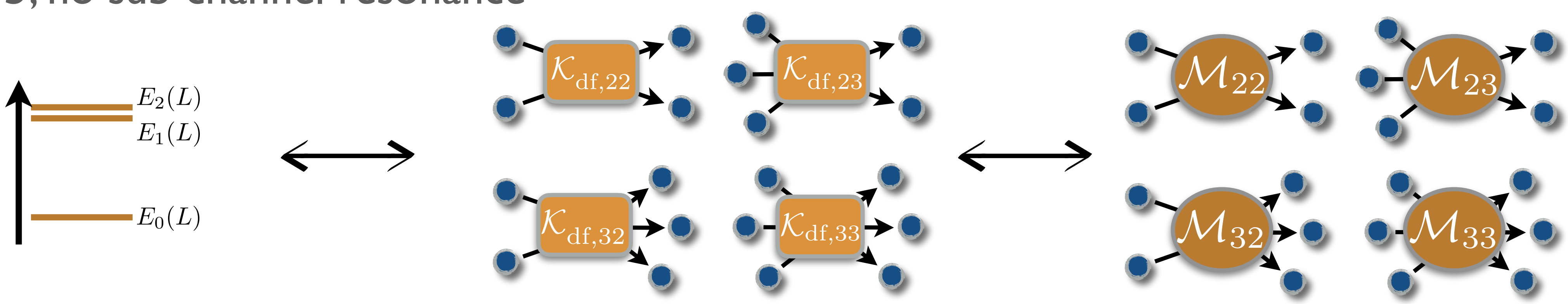
- General relation between ***energies*** and ***two-and-three scalar scattering***

No 2-to-3, ***no*** sub-channel resonance



• MTH, Sharpe (2014, 2015) •

2-to-3, no sub-channel resonance



• Briceño, MTH, Sharpe (2017) •

Including sub-channel resonances + ***different isospins*** + ***non-degenerate***

$$\pi\pi\pi \to \rho\pi \to \omega \to \rho\pi \to \pi\pi\pi$$

• Briceño, MTH, Sharpe (2018) • MTH, Romero-López, Sharpe (2020) • Blanton, Sharpe (2020)

Status…

- General relation between *energies* and *two-and-three scalar scattering*

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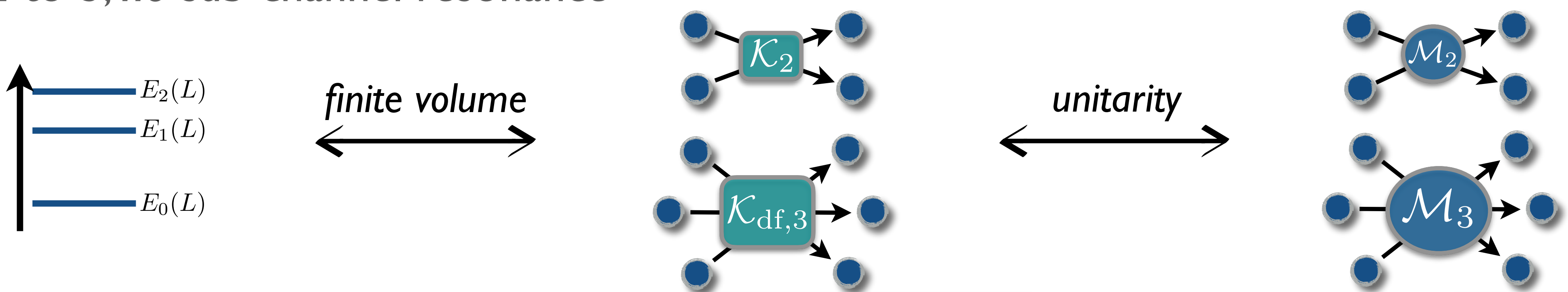

• MTH, Sharpe (2014, 2015) •

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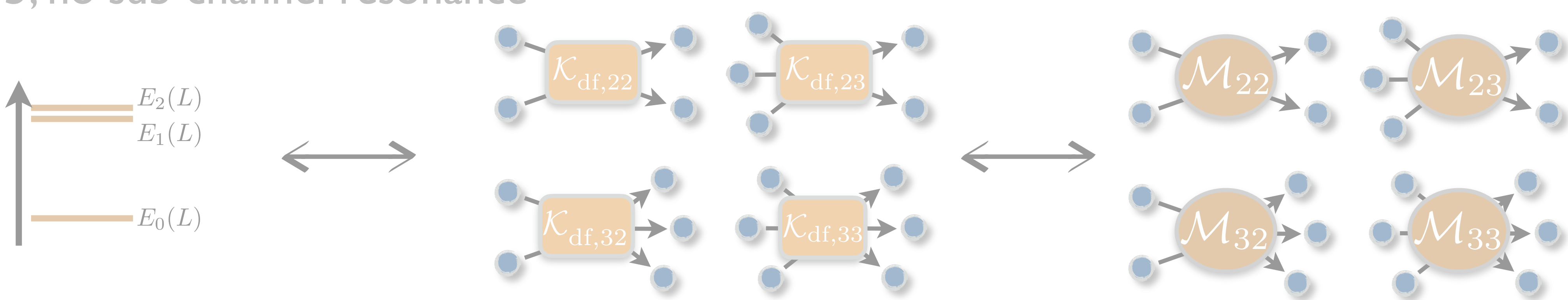


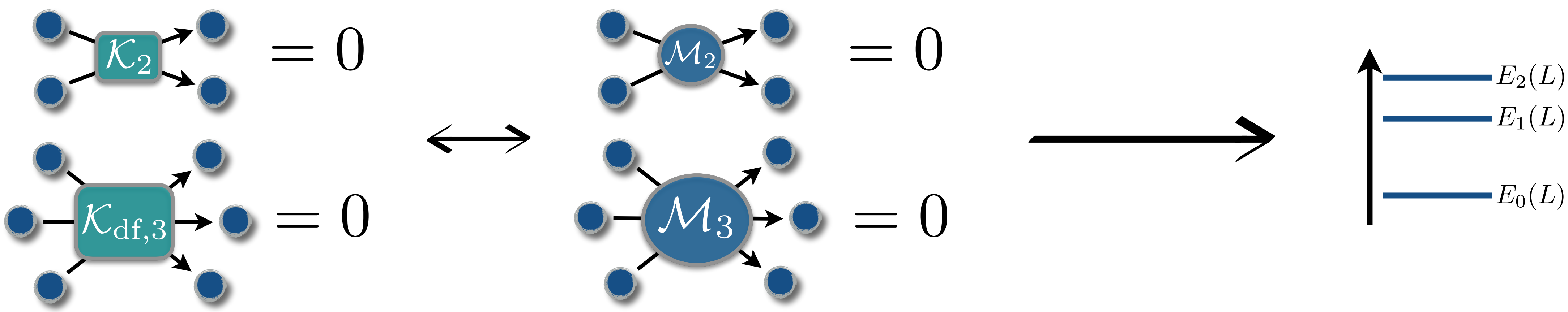
• Briceño, MTH, Sharpe (2017) •

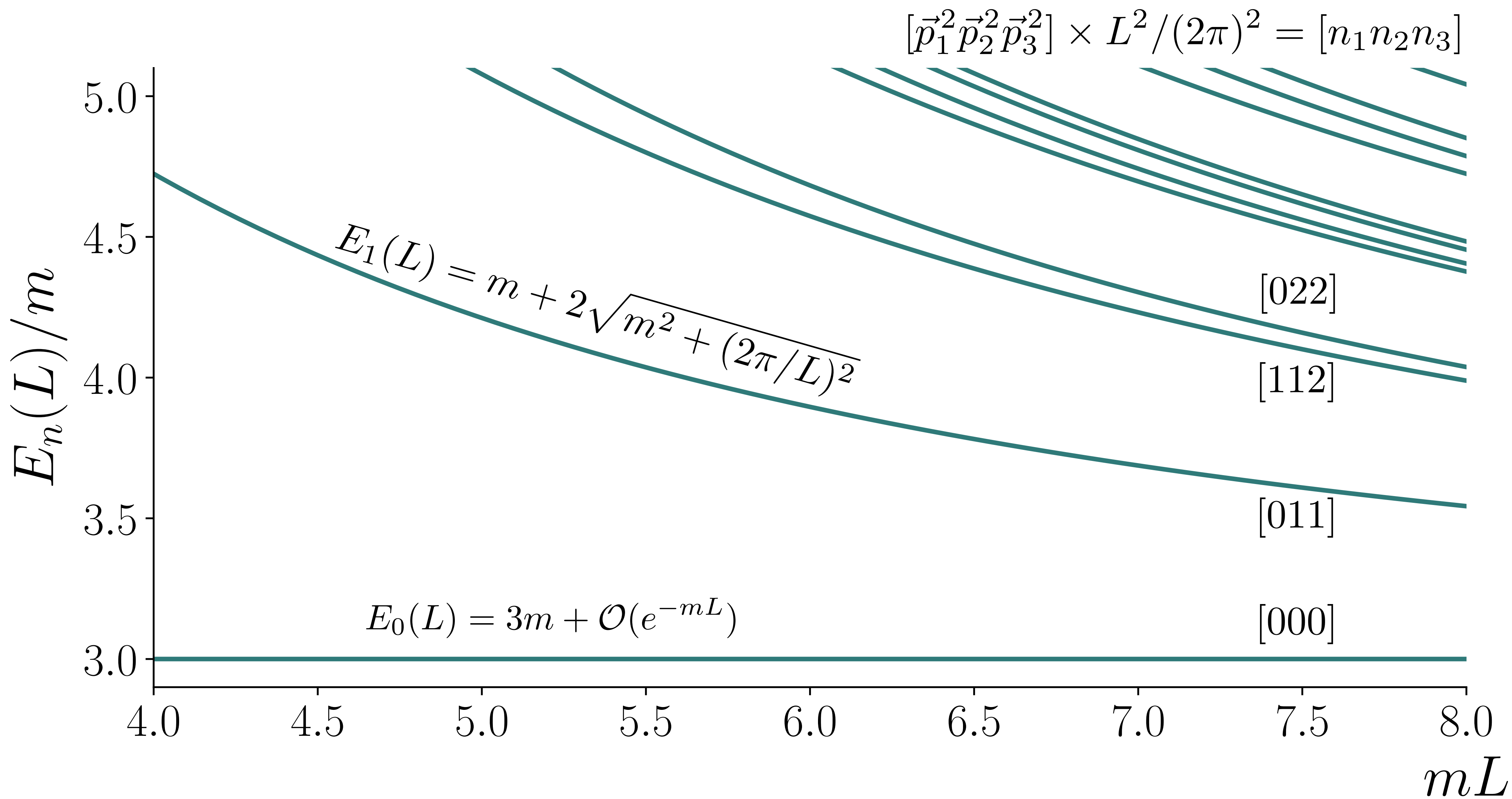
Including sub-channel resonances + *different isospins* + *non-degenerate*

$$\pi\pi\pi \to \rho\pi \to \omega \to \rho\pi \to \pi\pi\pi$$

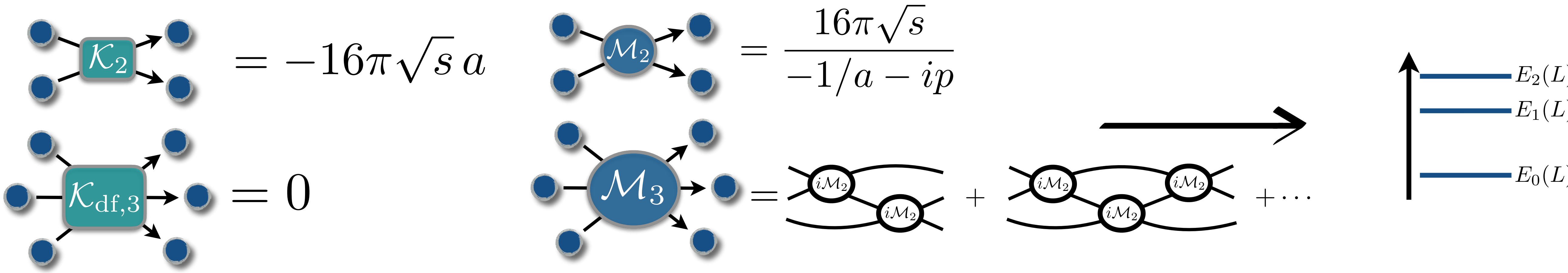
• Briceño, MTH, Sharpe (2018) • MTH, Romero-López, Sharpe (2020) • Blanton, Sharpe (2020)

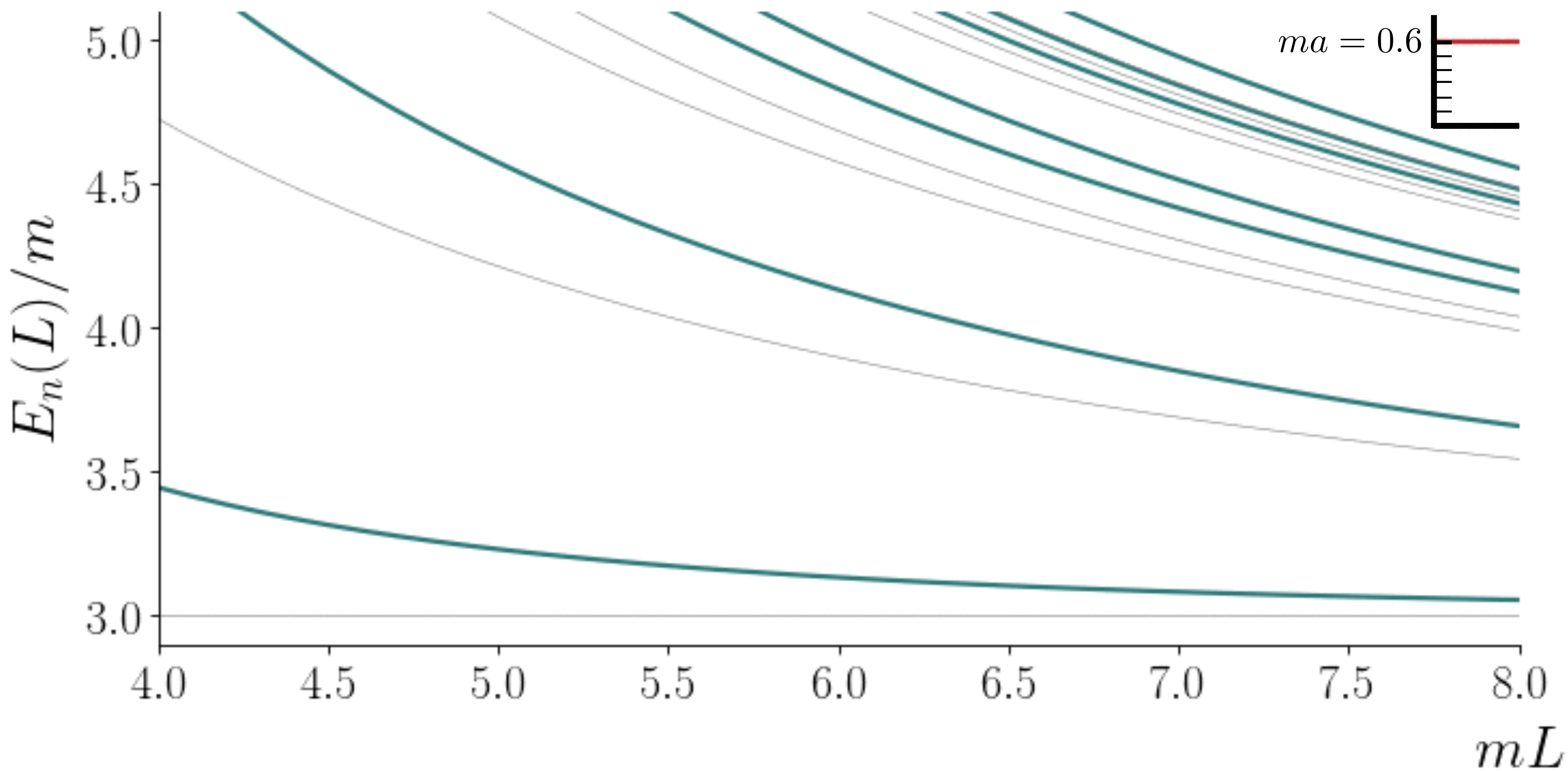
Non-interacting energies





Two-particle interactions





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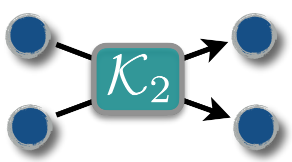
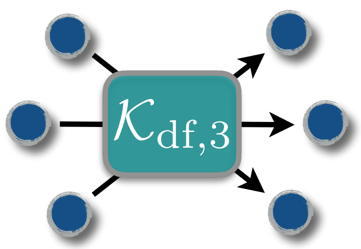
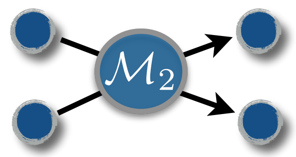


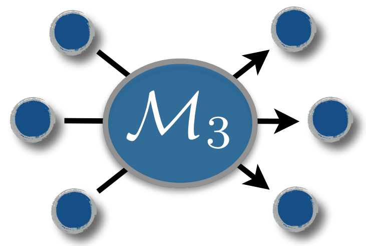
$= -16\pi\sqrt{s}\, a$



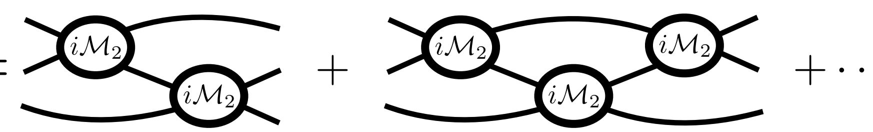
$= 0$

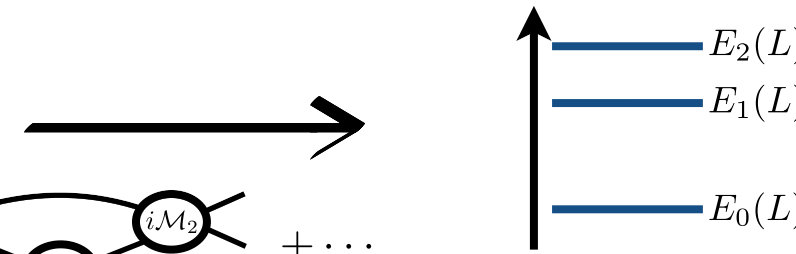


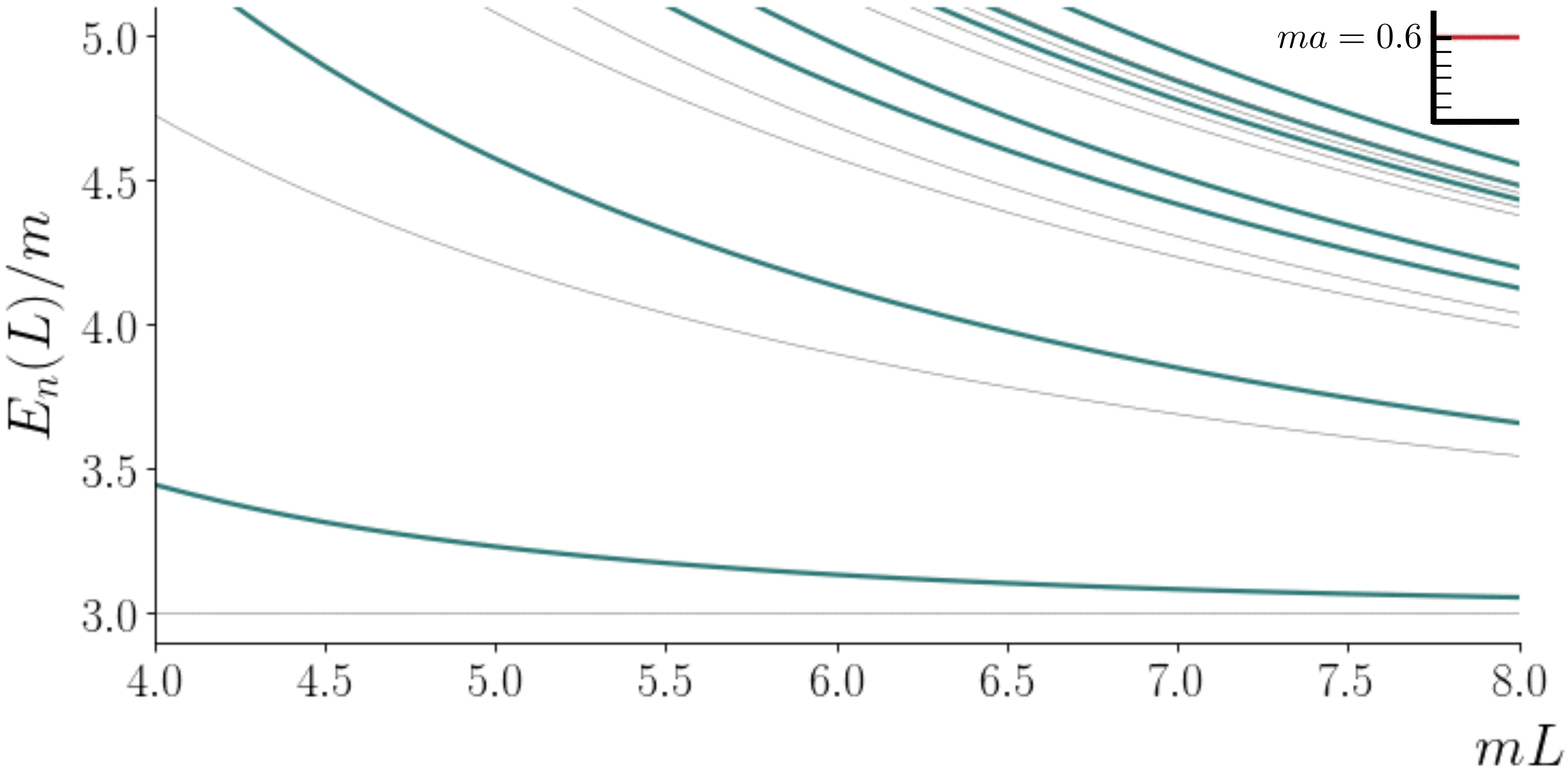
$$= \frac{16\pi\sqrt{s}}{-1/a - ip}$$



$=$

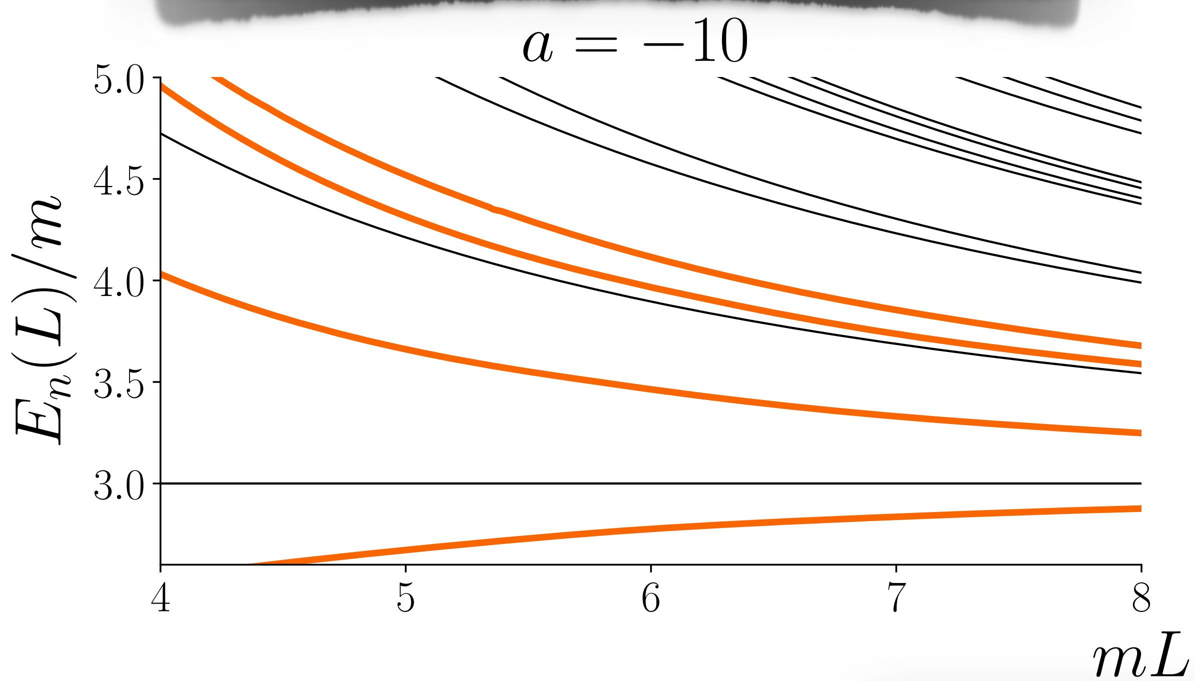




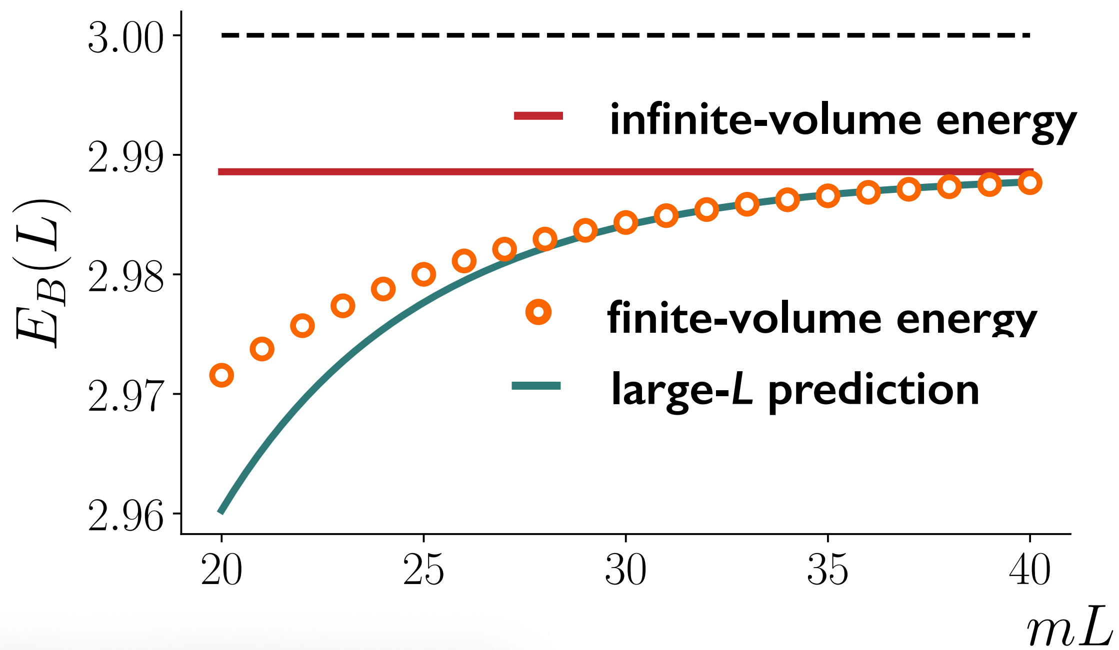


Many toy results

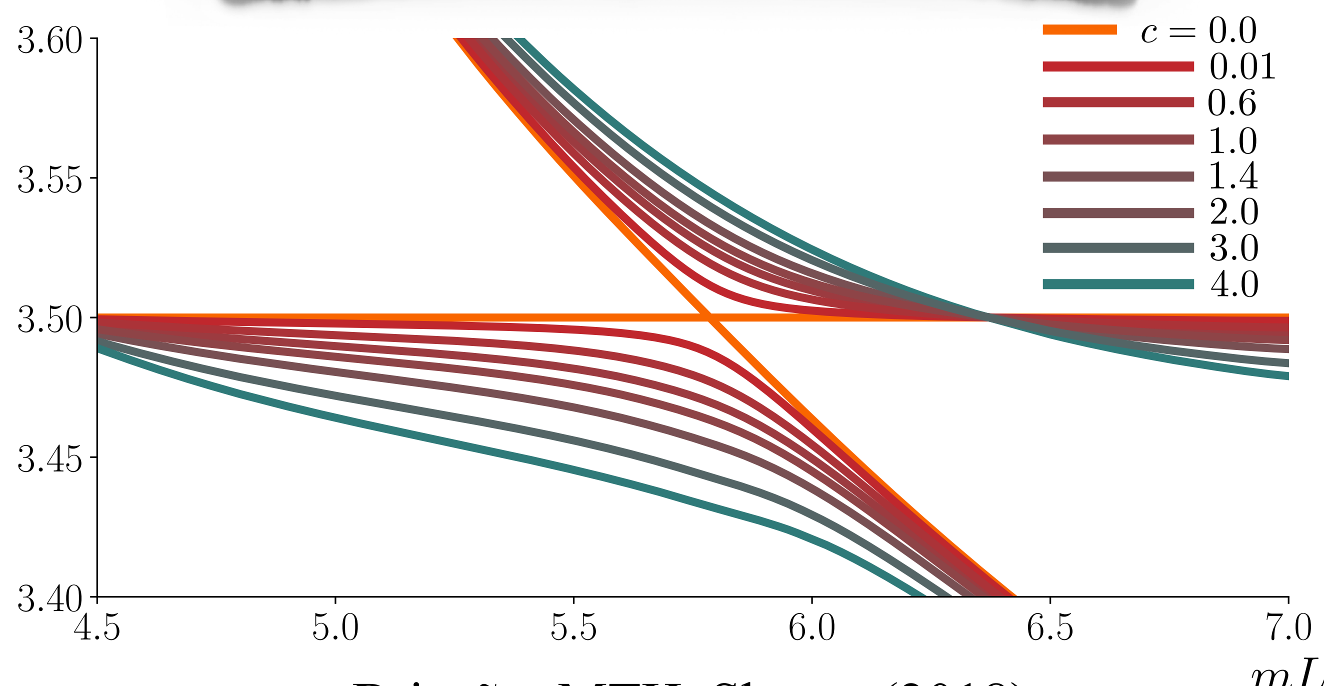
Spectrum with no 3-particle interaction



Finite-volume effects on a 3-particle bound state



Model of a 3-particle resonance



- Briceño, MTH, Sharpe (2018)

Related work

- Finite-volume unitarity method

  Döring, Mai (2016,2017)

  Gives connection to unitarity relations

- Non-relativistic EFT method

  Hammer, Pang, Rusetsky (2017)

  Simplified derivation + integral equations

Do not yet include non-identical and non-generate, angular momentum mixing, 2-to-3

- All methods

  Rely on intermediate, scheme-dependent quantity

  Hold up to e^{-mL} and for $E_3^\star < 5m_\pi$

  Equivalent where comparable

Review: *Lattice QCD and Three-particle Decays of Resonances*
MTH and Sharpe, *1901.00483*

Not covered here

- Activity extracting and fitting three-hadron energies

• Hörz, Hanlon •
• Blanton, Romero-López, Sharpe •
• Alexandru, Brett, Culver, Döring, Guo, Lee, Mai •
• Fischer, Kostrzewa, Liu, Romero-López, Ueding, Urbach •

- Activity connecting and extending formalisms

Relating infinite-volume equations • Jackura *et al.* (2019) •

Alternative derivations • Blanton, Sharpe (2020) •

Equivalence of formalisms (where comparable) • Blanton, Sharpe (2020) •

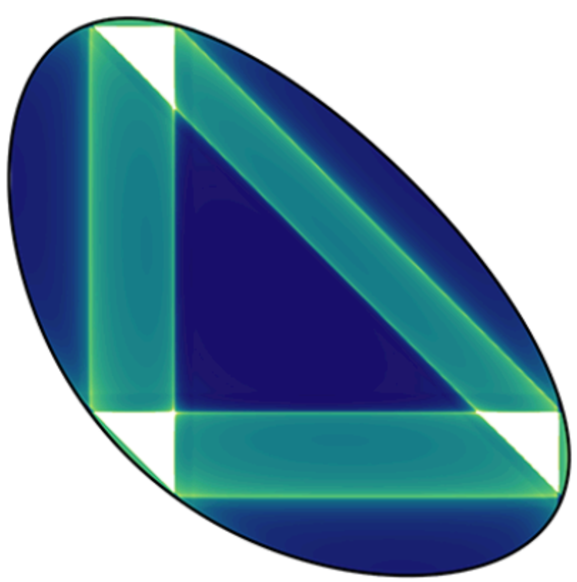
Three-hadron decays • Müller, Rusetsky (2021) • MTH, Romero-López, Sharpe (2021) •

Energy-Dependent $\pi^+\pi^+\pi^+$ Scattering Amplitude from QCD

Maxwell T. Hansen[1,2,*] Raul A. Briceño,[3,4,†] Robert G. Edwards,[3,‡]
Christopher E. Thomas,[5,§] and David J. Wilson[5,‖]

(for the Hadron Spectrum Collaboration)



EDITORS' SUGGESTION

Energy-Dependent $\pi^+ \pi^+ \pi^+$ Scattering Amplitude from QCD

A three-hadron scattering amplitude is computed using lattice QCD for the first time.

Maxwell T. Hansen *et al.*

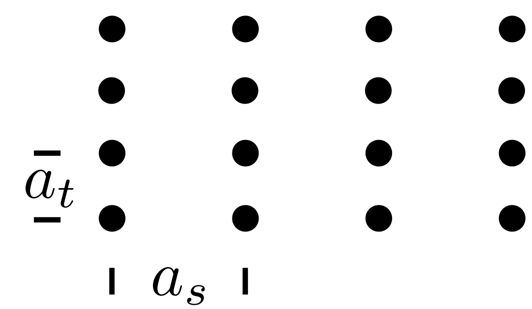
Phys. Rev. Lett. **126**, 012001 (2021)

$$\pi^+\pi^+\pi^+ \to \pi^+\pi^+\pi^+$$

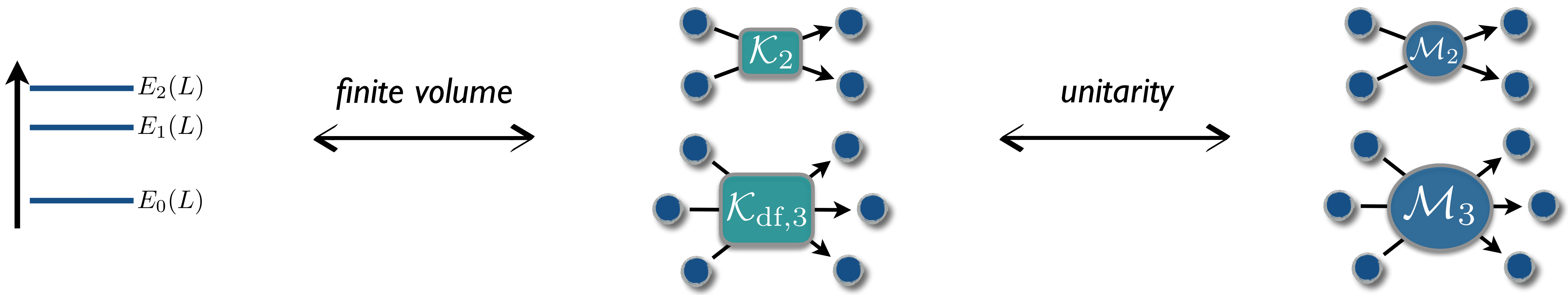
lattice details

$N_f = 2 + 1$ $\quad a_s/a_t = 3.444(6)$ $\quad L_s/a_s = 20, 24$

$m_\pi \approx 400\text{MeV}$ $\quad a_s \approx 0.12\text{fm}$



☐ Workflow outline



MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

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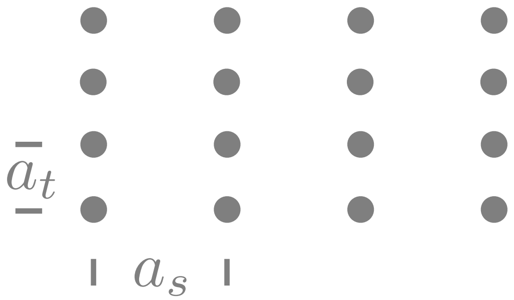

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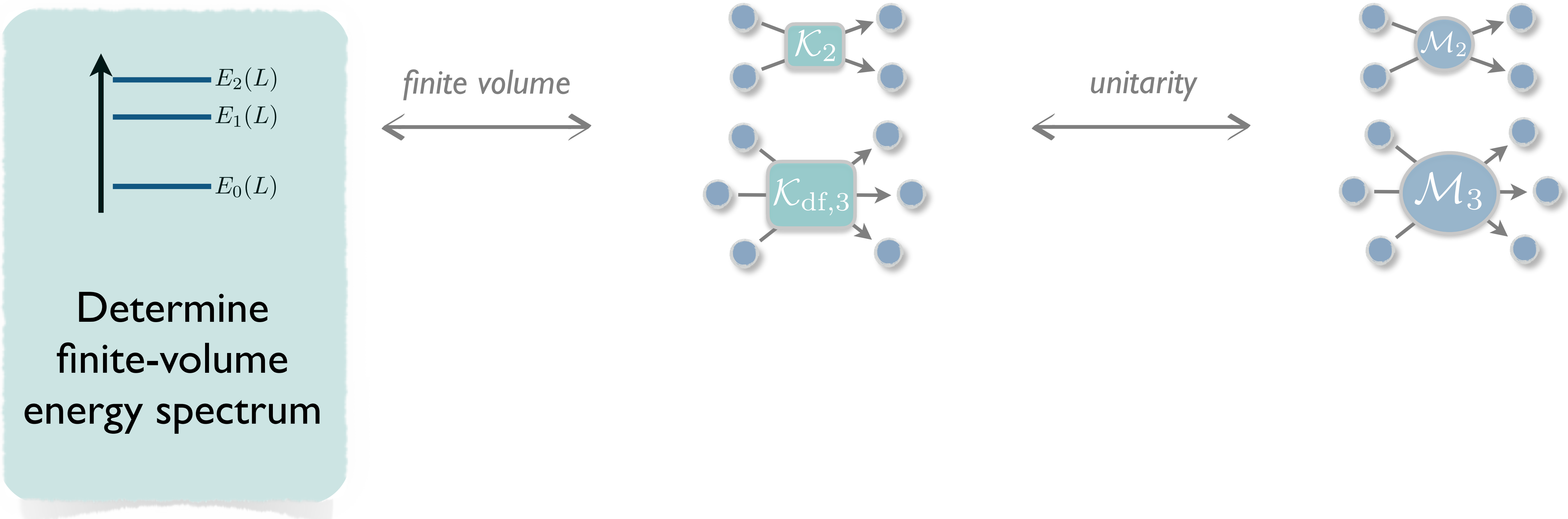
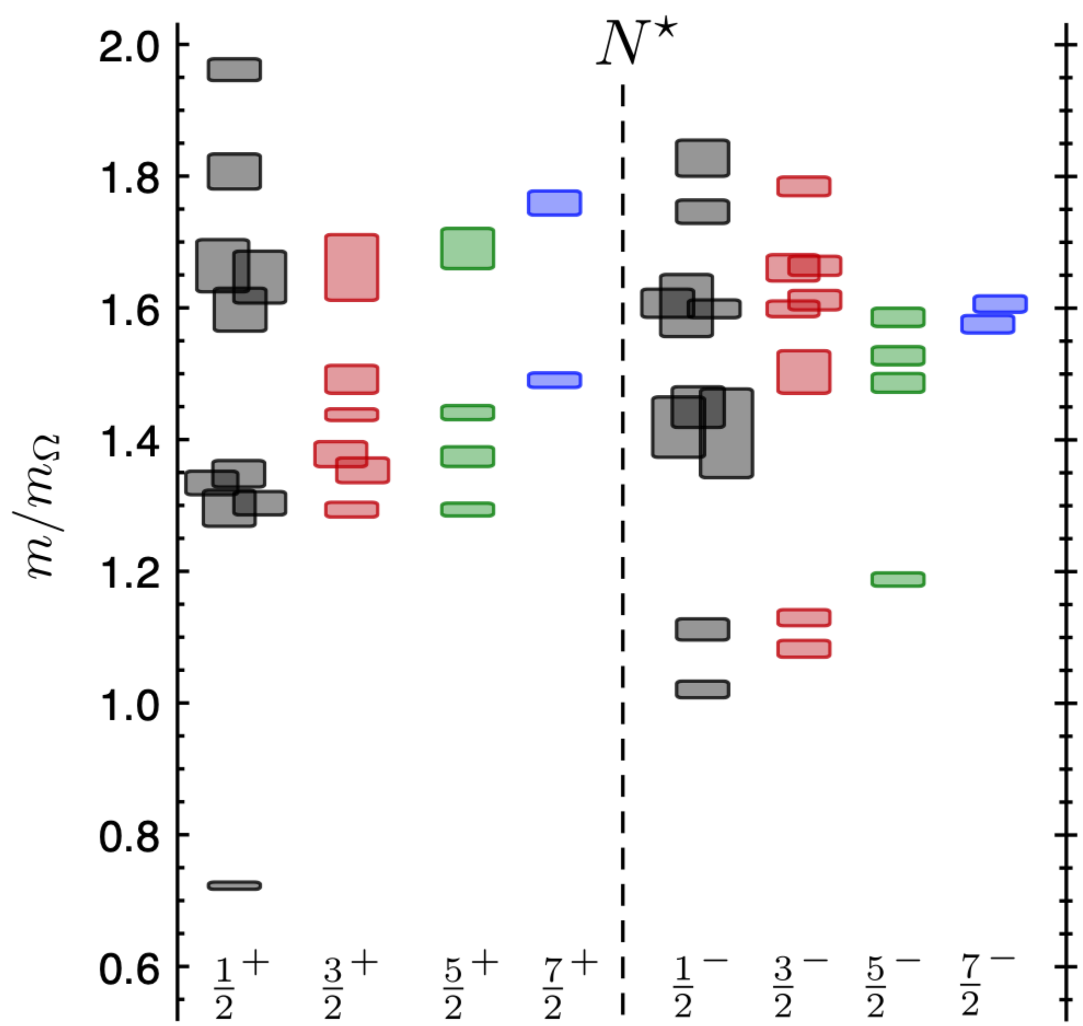


MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

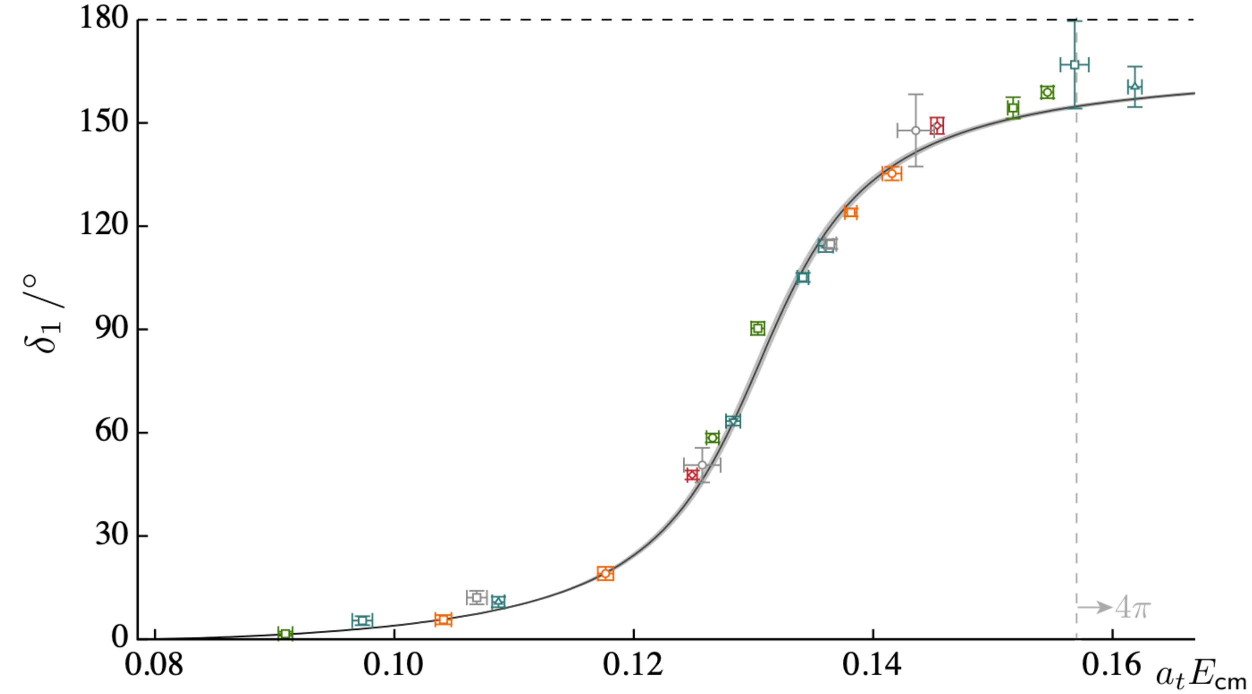
Two types of spectroscopy

Explore the spectrum of compact QCD excited states (via quark-model inspired operators)



Edwards, Dudek, Richards, Wallace (2011)

Extract the honest finite-volume energy spectrum



Wilson, Briceño, Dudek, Edwards, Thomas (2015)

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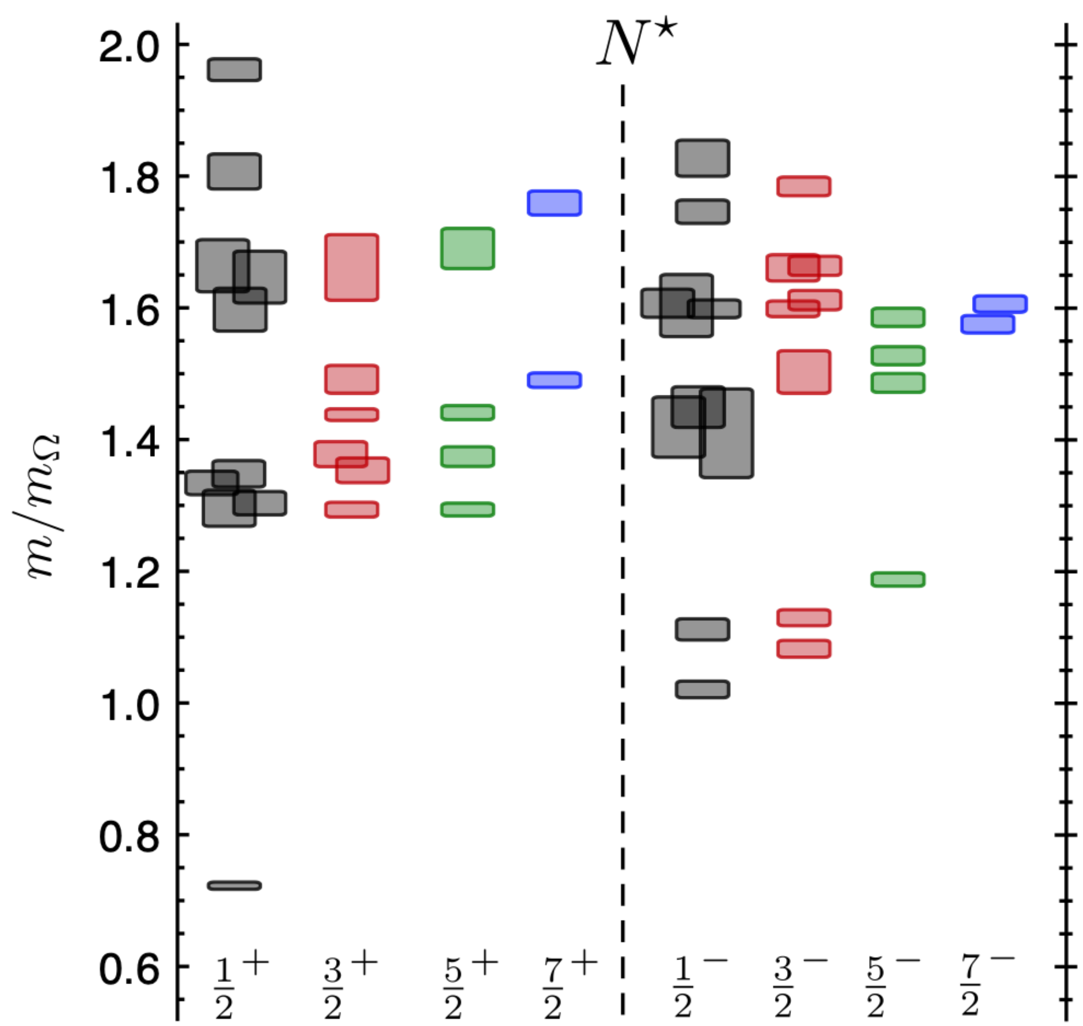
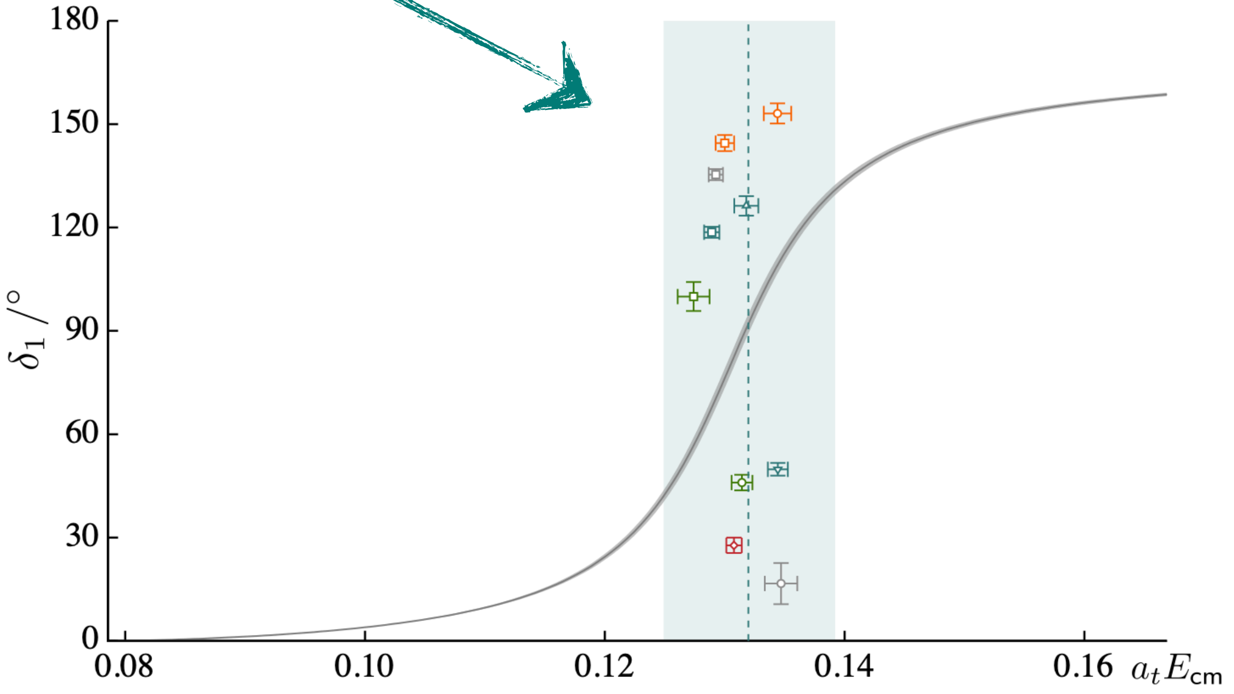


local operator spectrum =
not suitable for phase shift extraction



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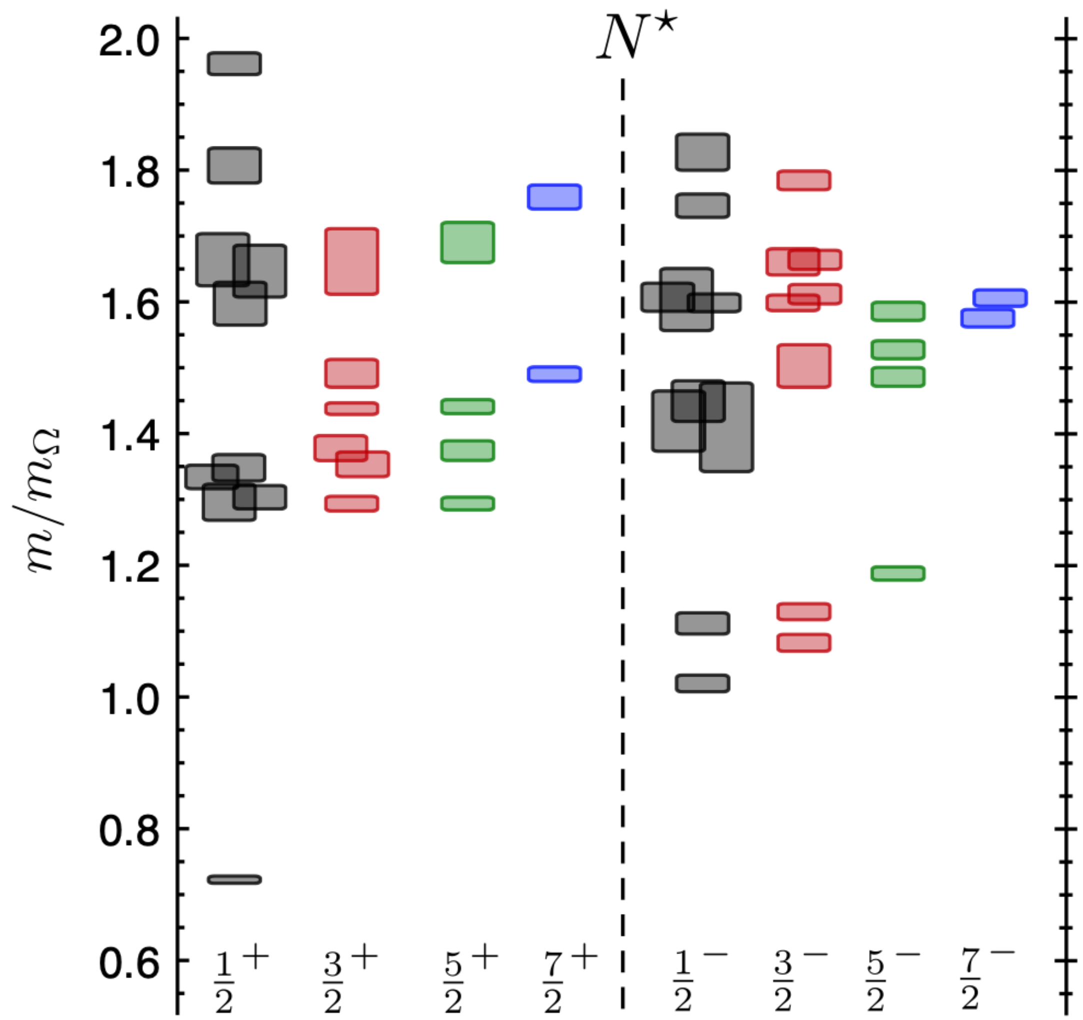
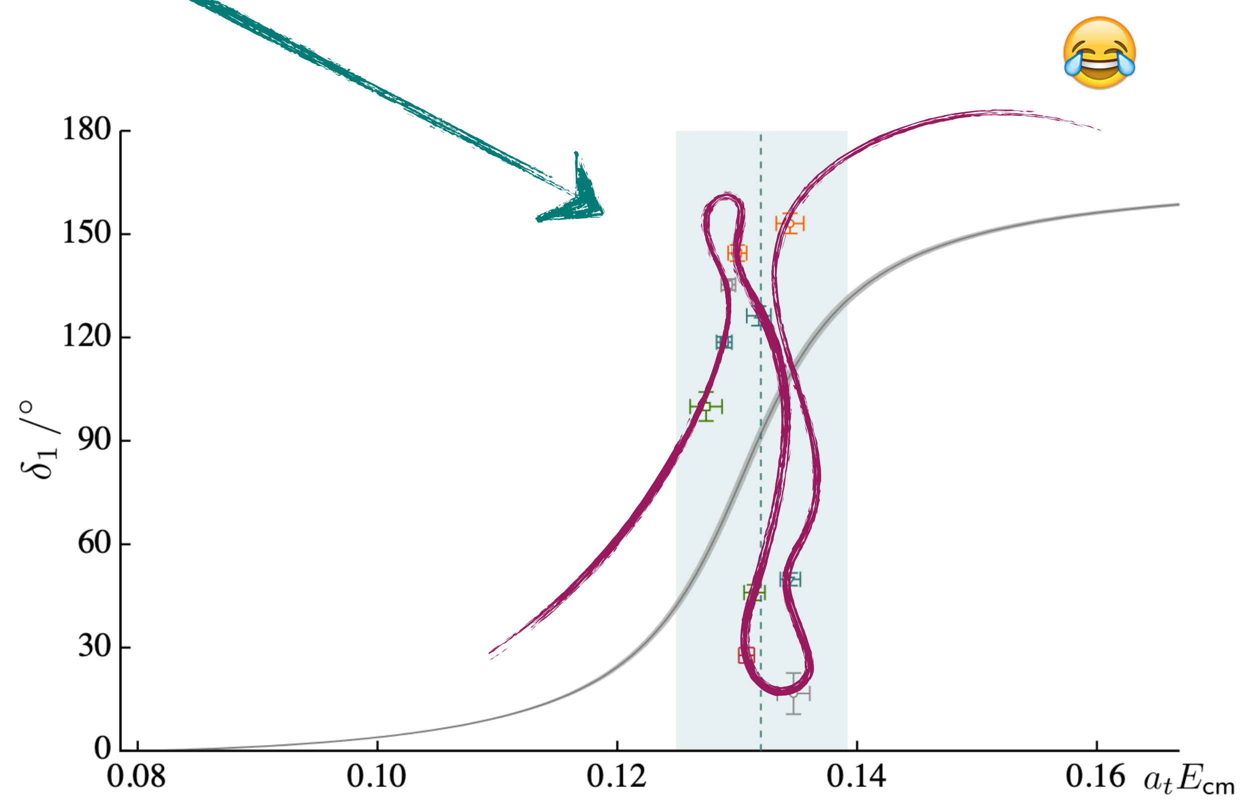




Edwards, Dudek, Richards, Wallace (2011)

Wilson, Briceño, Dudek, Edwards, Thomas (2015)

Extracting the finite-volume spectrum

- Derivatives + gamma matrices + smearing → *basis of single-hadron operators*

$$\overline{q}\Gamma q, \quad \overline{q}\Gamma D q, \quad \overline{q}\Gamma D \cdots D q$$

- Variational method → *optimized single hadron*

$$\pi = c_1\, \overline{q}\Gamma q + c_2\, \overline{q}\Gamma D q + \cdots$$

- Group theory + individual momentum projection → *two- and three-pion operators*

$$(\pi\pi\pi)(\boldsymbol{P}, \Lambda) = \sum \mathrm{CG}\, \pi(\boldsymbol{p}_1)\pi(\boldsymbol{p}_2)\pi(\boldsymbol{p}_3)$$

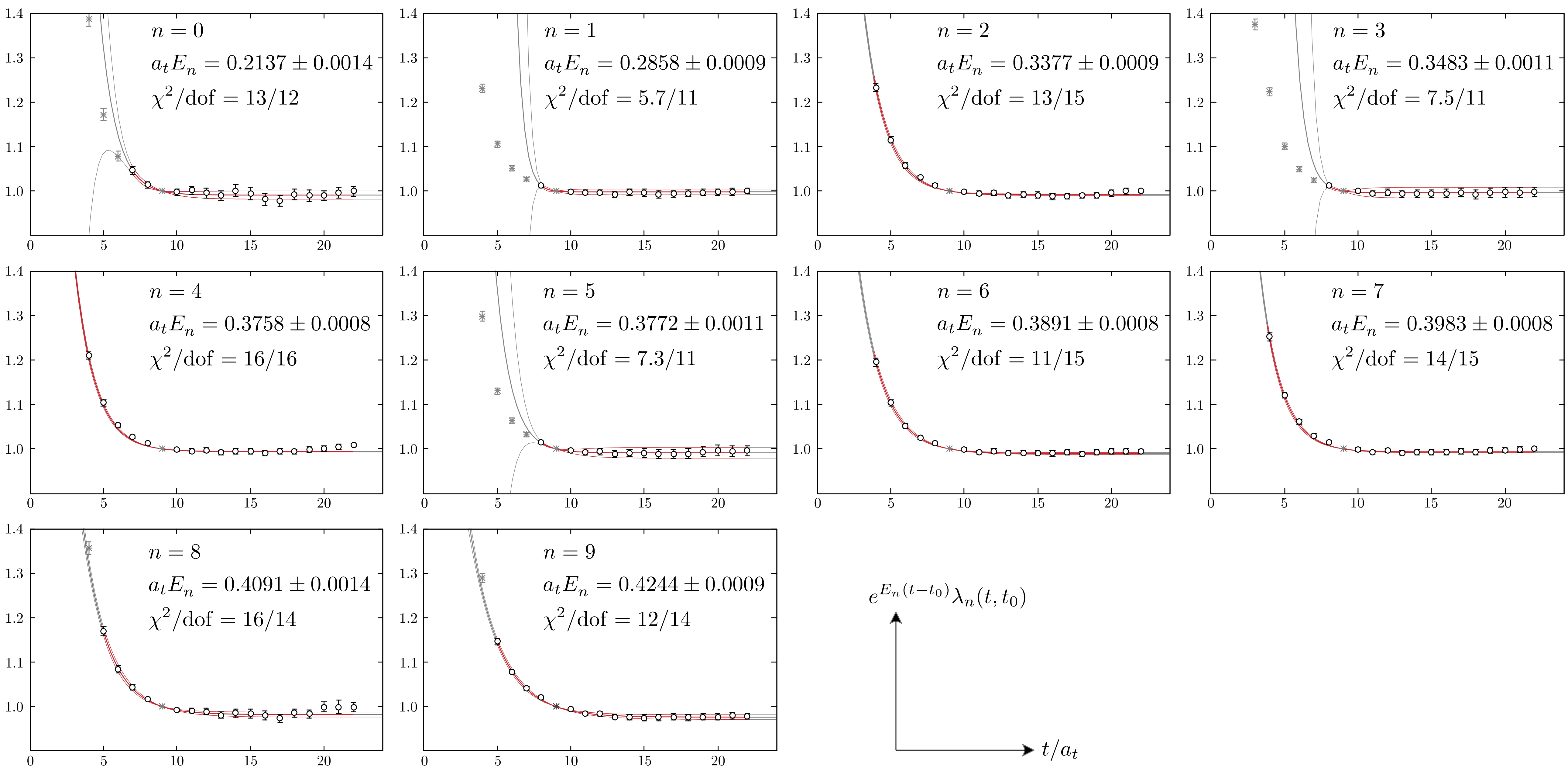
- Second variational method → *multi-pion finite-volume energies*

- Validate extraction…
  - Quality of energy plateaus
  - Stability under change of operators
  - Consistent with finite-volume formalism

Brought to you by distillation!

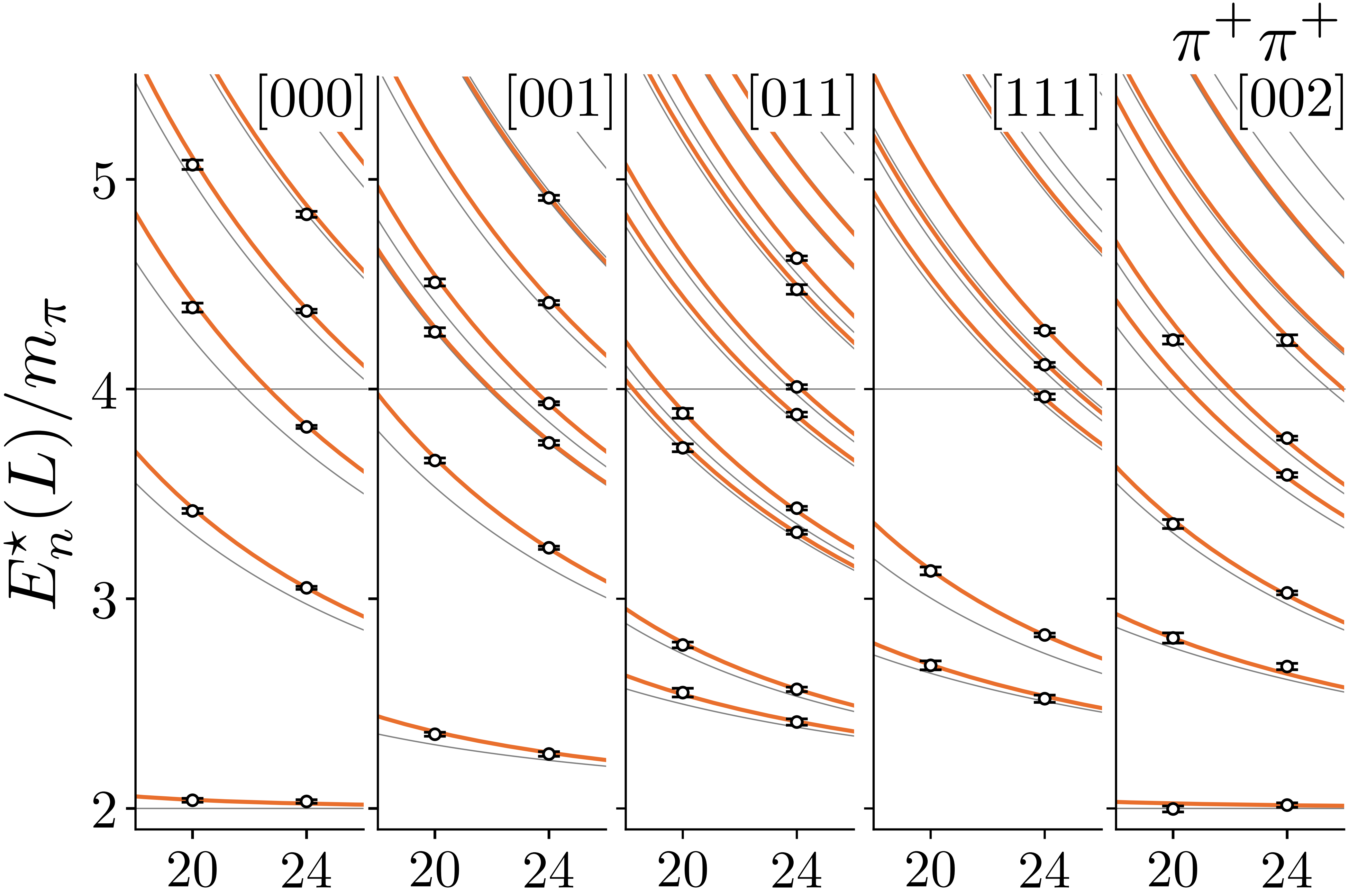
Peardon *et al.* (2009)

$I = 3\,(\pi^+\pi^+\pi^+),\ \boldsymbol{P} = [000],\ \Lambda = A_1^-,\ L/a_s = 24$



MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

$\pi^+\pi^+$ energies



MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

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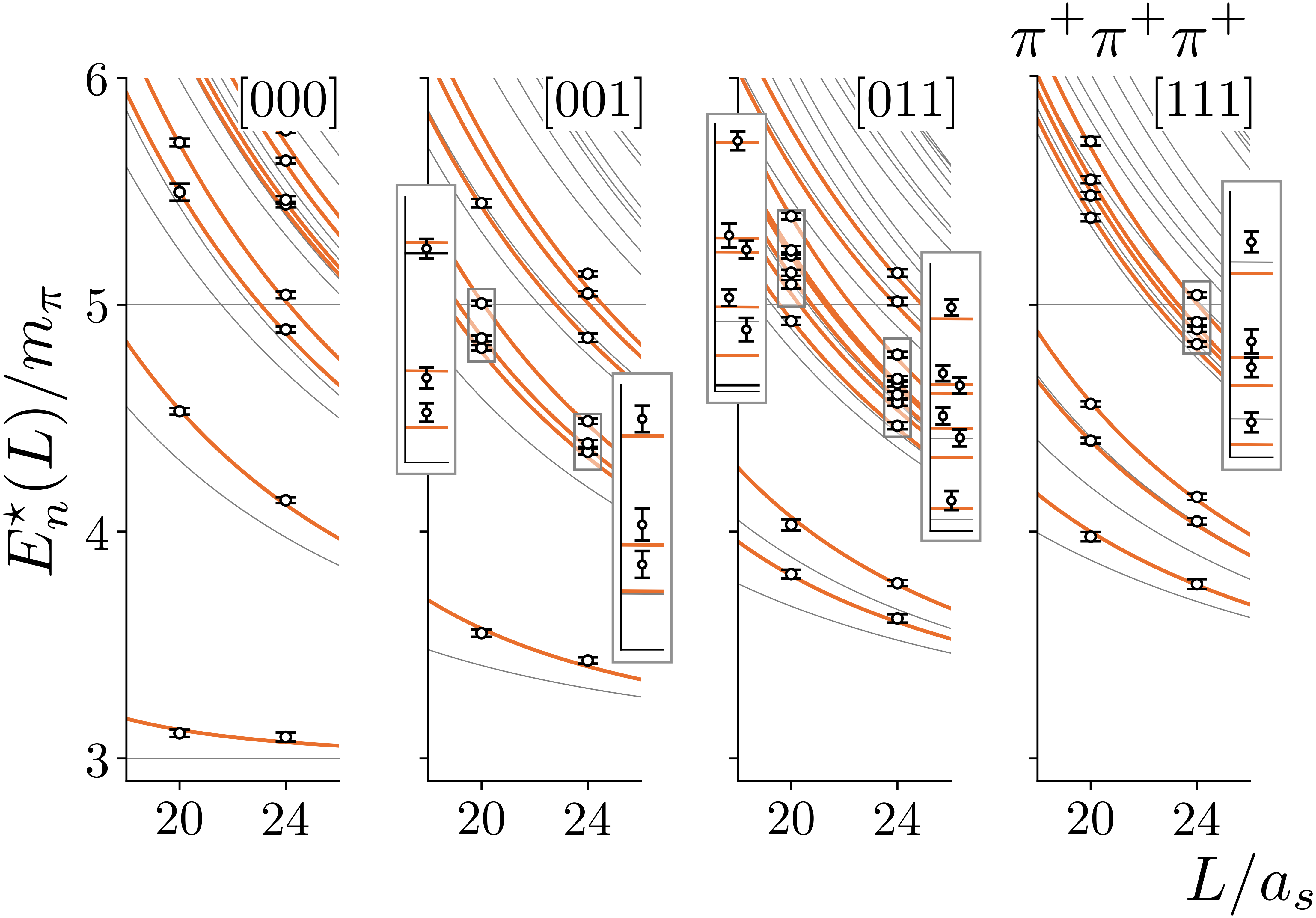

MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

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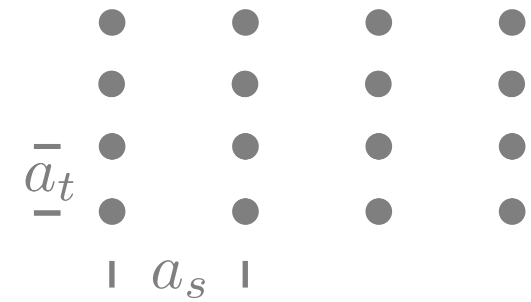
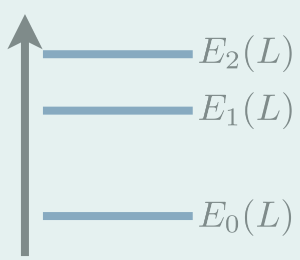


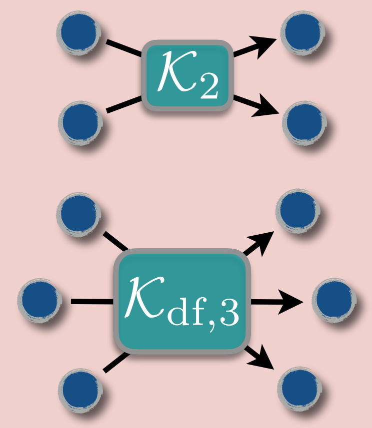
- [ ] Workflow outline



Determine finite-volume energy spectrum

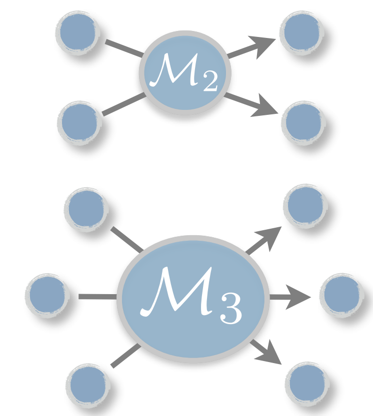
✓

finite volume



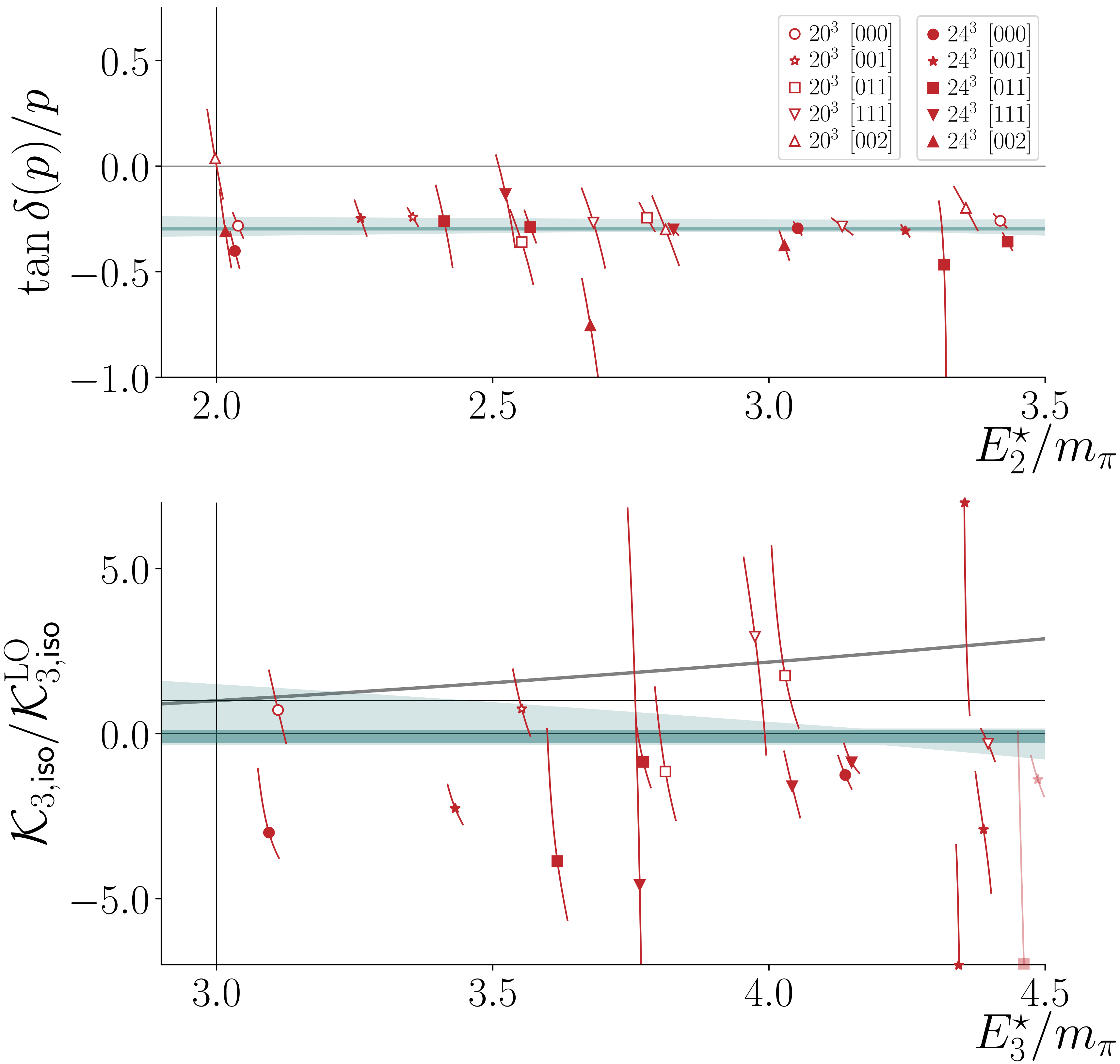
Fit to constrain two- and three-body K matrices

unitarity



MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

K matrix fits



Finite-volume formalism relates energies to K matrices

One-to-one for $K_{\text{df},3}$ depending only on $E_{\text{cm}} = E^\star$

Fit both two and three-body K to various polynomials

Cut on the CM energy in the fits

$K_{\text{df},3}$ is scheme dependent (removed upon converting to $\mathcal{M}_3$)

$\pi^+\pi^+\pi^+ \to \pi^+\pi^+\pi^+$

lattice details

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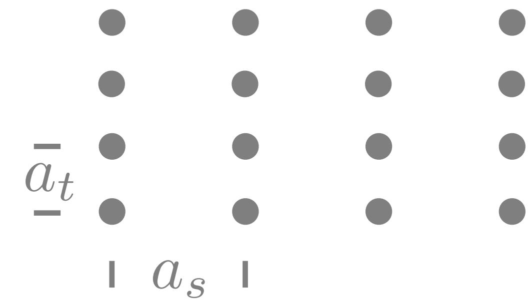
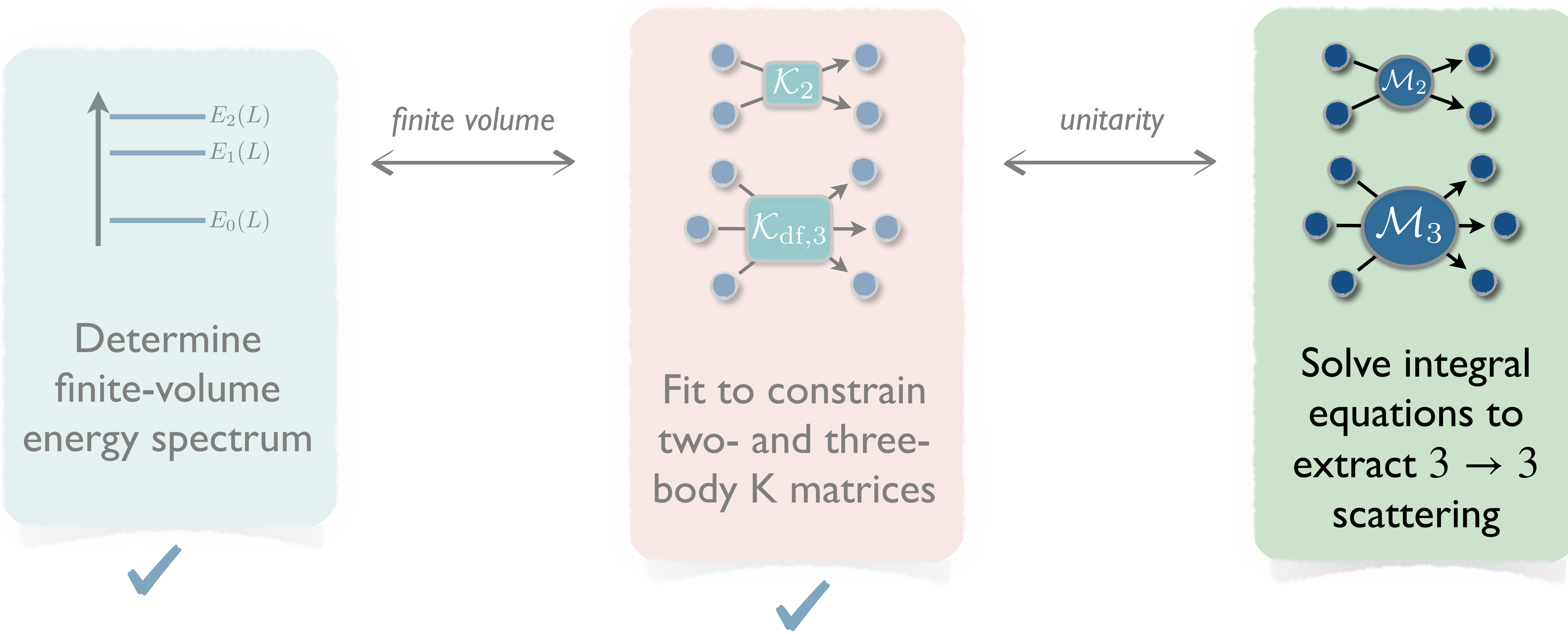


Workflow outline



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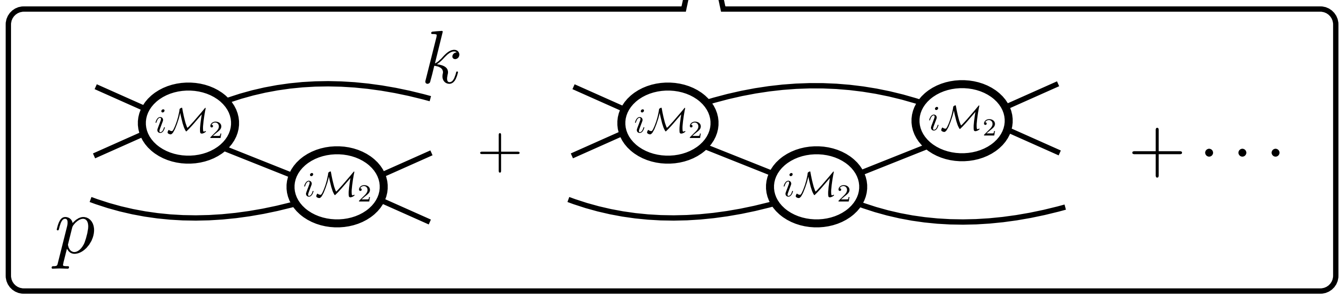
Integral equation

$$\mathcal{M}_3^{\mathsf{un}}(E_3^\star, \boldsymbol{p}, \boldsymbol{k}) = \mathcal{D}^{\mathsf{un}}(E_3^\star, \boldsymbol{p}, \boldsymbol{k}) + \mathcal{E}^{\mathsf{un}}(E_3^\star, \boldsymbol{p})\mathcal{T}(E_3^\star)\mathcal{E}^{\mathsf{un}}(E_3^\star, \boldsymbol{k})$$



Vanishes for $K_{\text{df},3} = 0$

$$\boldsymbol{D}(N,\epsilon) = -\boldsymbol{\mathcal{M}} \cdot \boldsymbol{G}(\epsilon) \cdot \boldsymbol{\mathcal{M}} - \boldsymbol{\mathcal{M}} \cdot \boldsymbol{G}(\epsilon) \cdot \boldsymbol{P} \cdot \boldsymbol{D}(N,\epsilon)$$

$$\mathcal{D}^{\mathsf{un}}(E_3^\star, \boldsymbol{p}, \boldsymbol{k}) = \lim_{\epsilon\to 0} \lim_{N\to\infty} \boldsymbol{D}_{pk}(N,\epsilon)$$

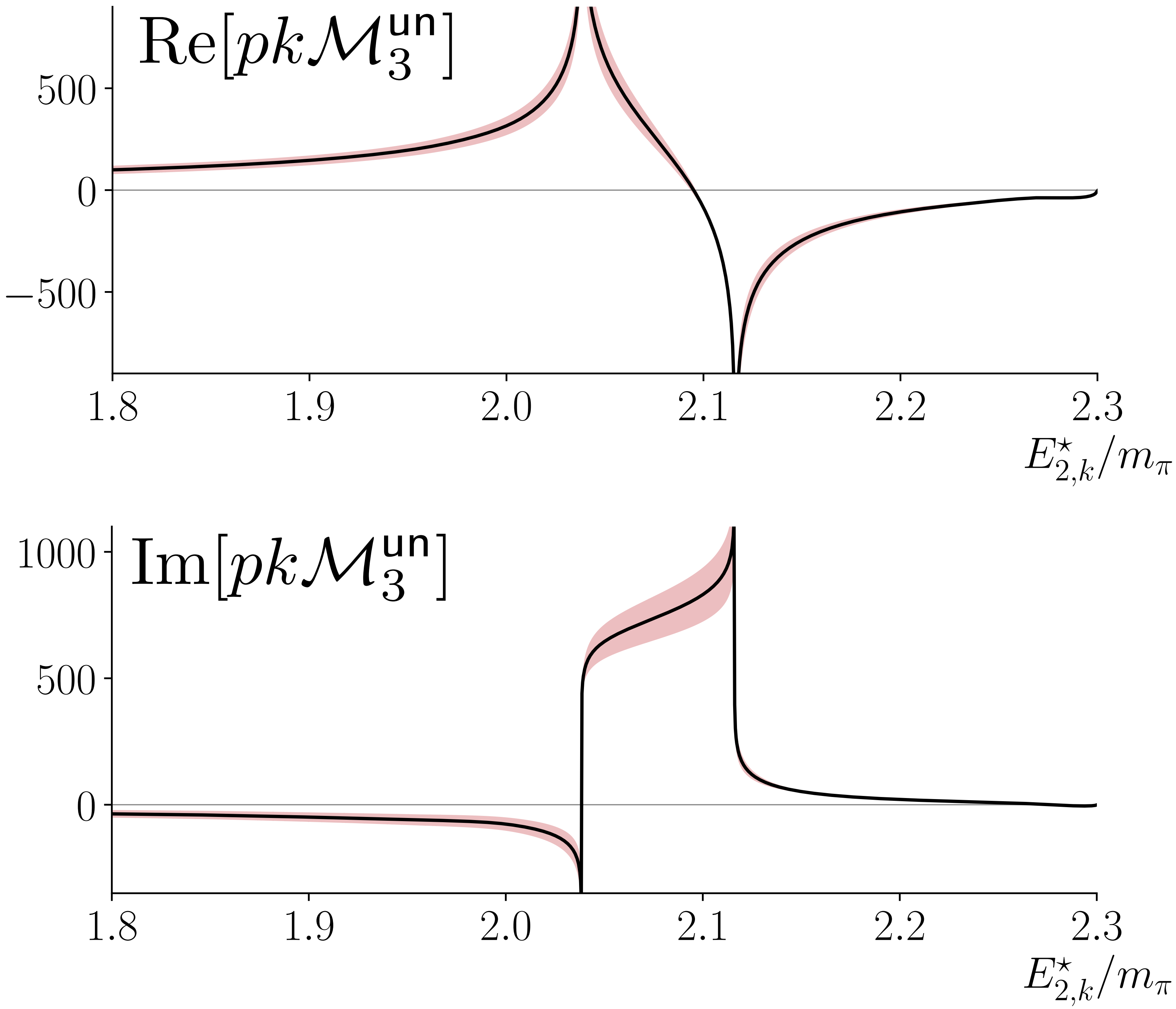
- See also…

Solving relativistic three-body integral equations in the presence of bound states

Andrew W. Jackura,[1,2,*] Raúl A. Briceño,[1,2,†] Sebastian M. Dawid,[3,4,‡] Md Habib E Islam,[2,§] and Connor McCarty[5,¶]

arXiv: 2010.09820

Integral equation



Total angular momentum = 0

Two-particle sub-system angular momentum = 0

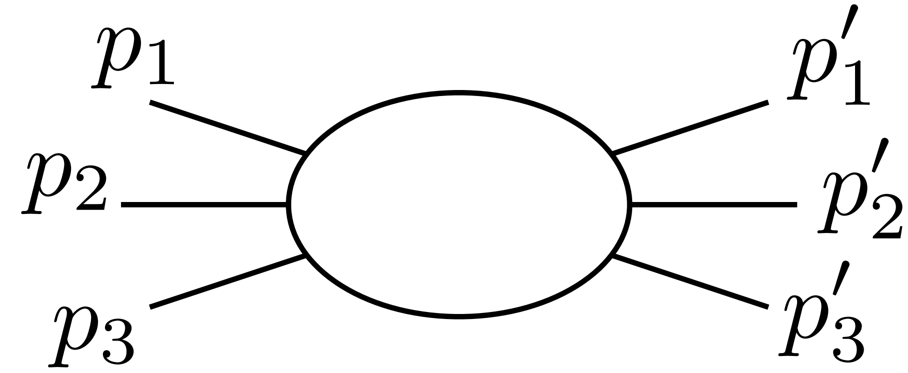
Plot at fixed $E_3^\star$ and p

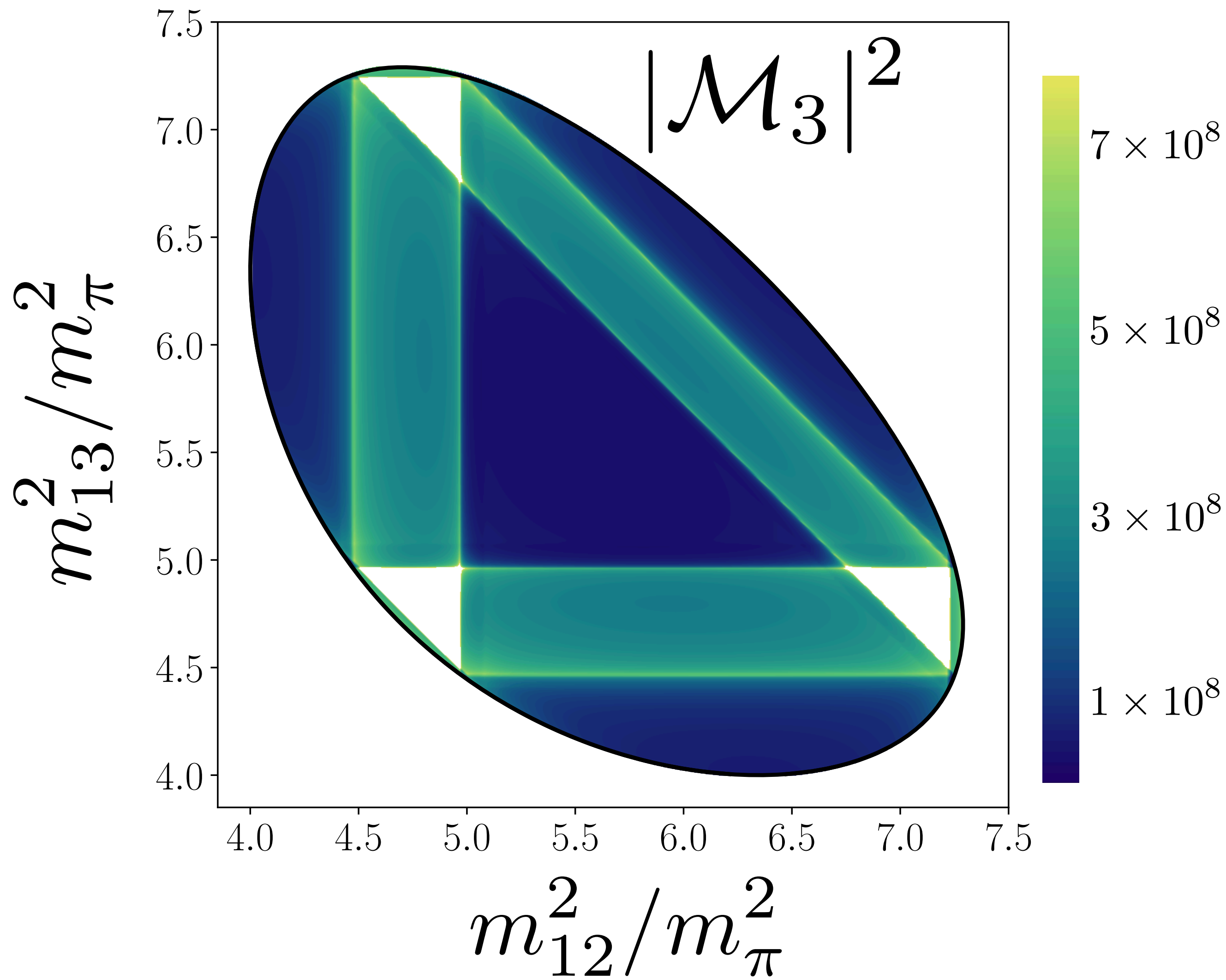
Both two- and three-body uncertainties estimated

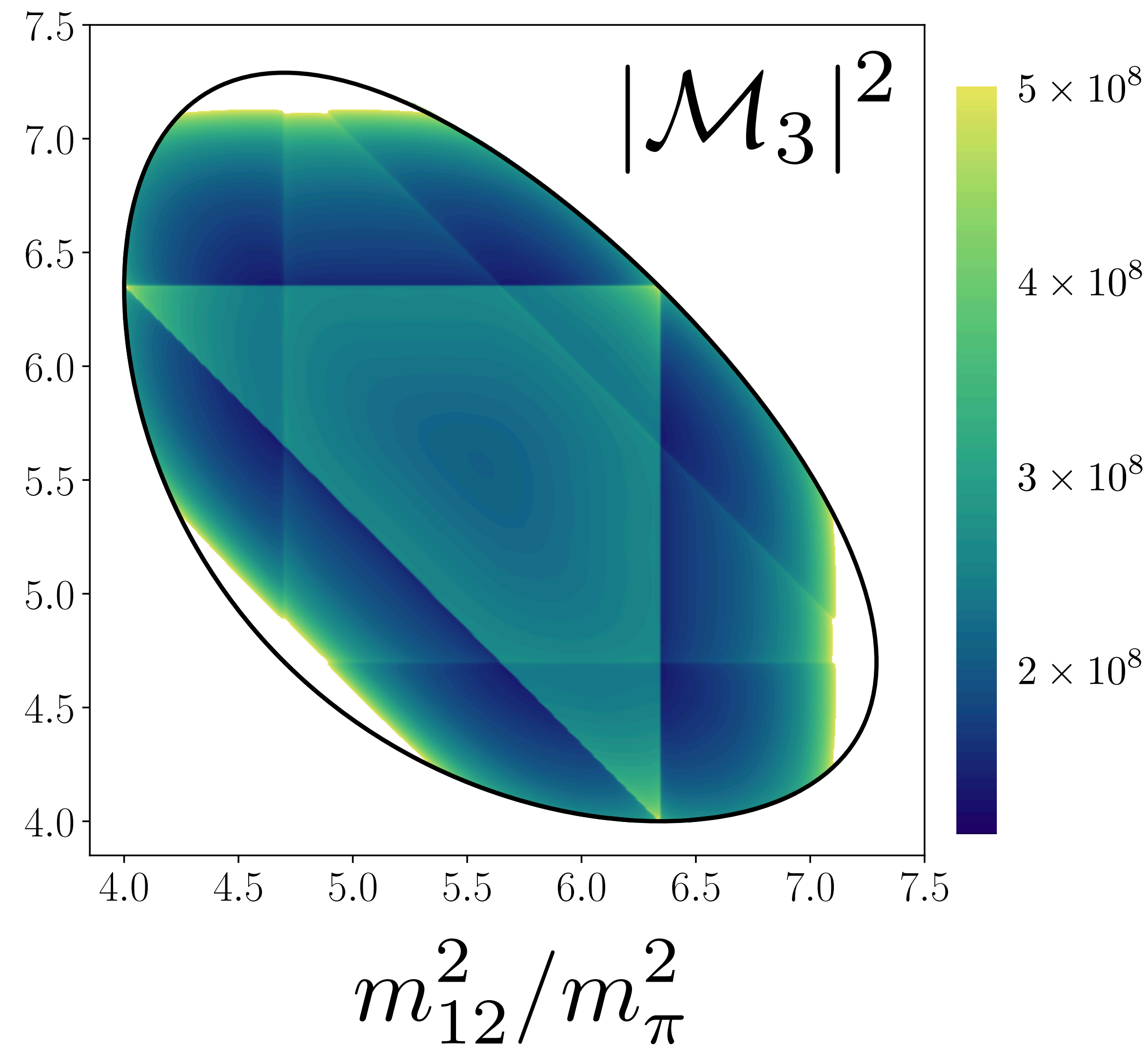
Still need to symmetrize

MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

$$\mathcal{M}_3 = \sum_{i,j \in \{1,2,3\}} \mathcal{M}_3^{\mathsf{un}}(p_i', p_j)$$







MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

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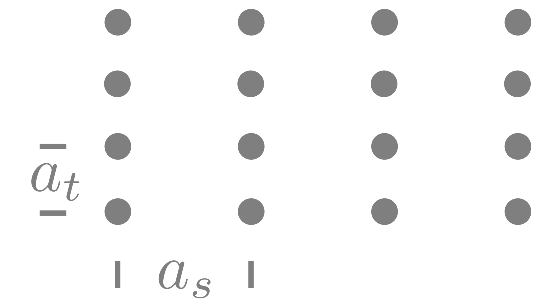
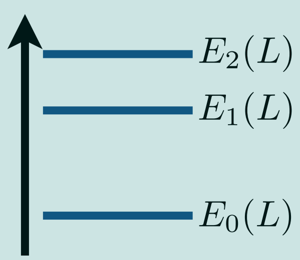
lattice details

$N_f = 2+1$ $\quad a_s/a_t = 3.444(6)$ $\quad L_s/a_s = 20, 24$

$m_\pi \approx 400\text{MeV}$ $\quad a_s \approx 0.12\text{fm}$



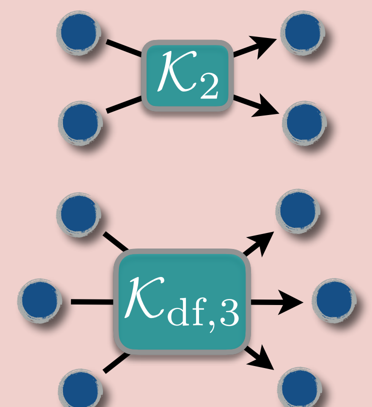
- [ ] Workflow outline



Determine finite-volume energy spectrum

✓

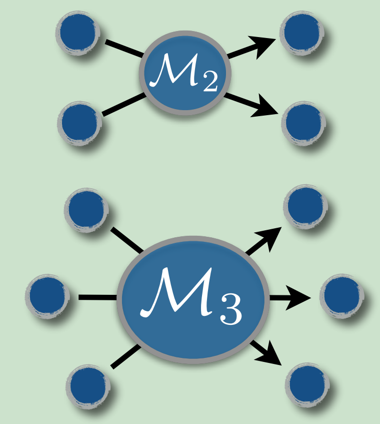
finite volume ⟷



Fit to constrain two- and three-body K matrices

✓

unitarity ⟷



Solve integral equations to extract $3 \to 3$ scattering

✓

MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

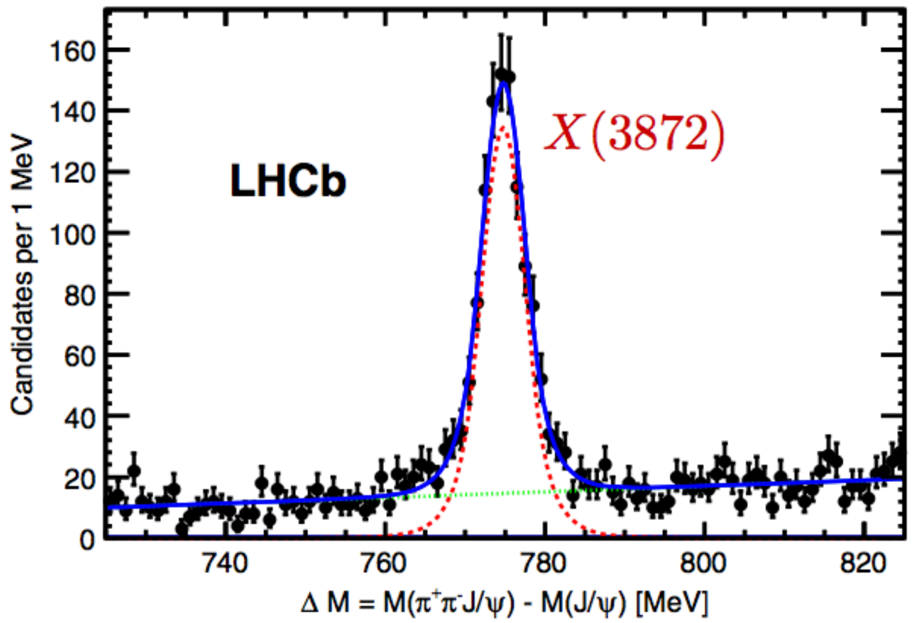
Outlook for exotics

- Challenging for many reasons

incredible hierarchy of scales

$$M_X - M_{D^0 D^{0*}} = 0.01 \pm 0.18\,\mathrm{MeV}$$

$$\Gamma(X) < 1.2\mathrm{MeV}$$



- Lebed, Mitchell, Swanson, *PPNP review* (2017) •
- LHCb (PRD92, 2015) •

many open channels $X \to \omega J/\psi,\ \pi\pi J/\psi,\ D^{*0}\bar{D}^0,\ \cdots$

- My speculative outlook

should be possible to derive N-particle volume formalism

then relate K-matrices to spectra to identify LQCD strategy

hierarchy of scales = not always a problem $\mathcal{K}_2 \propto a \to \infty$

Conclusions

- LQCD is in the era of 'rigorous resonance spectroscopy'
- The finite-volume = ***a useful tool***
- Challenges and progress

  formal analysis was technical → *ground work is now set*

  scattering demands high precision excited states → *advanced algorithms make this possible*

  many calculations at unphysical quark masses → *physical-mass scattering now appearing*

  → *varying masses probes resonance structure*

  3-body amplitude is highly singular → *intermediate K matrix is not*

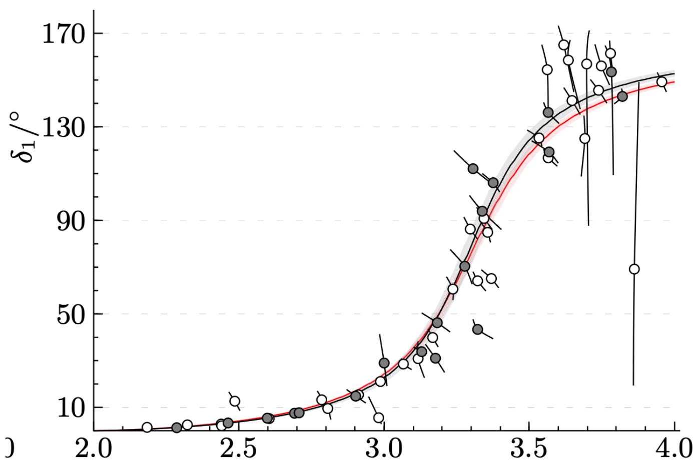
- Next steps…

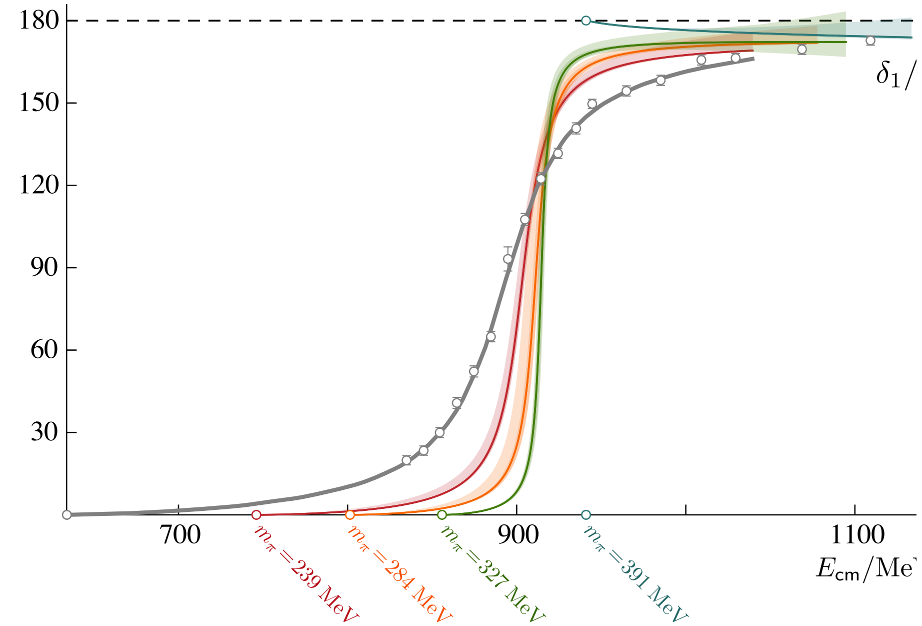
  complete 3-particle formalism → *extend to N-particle formalism*

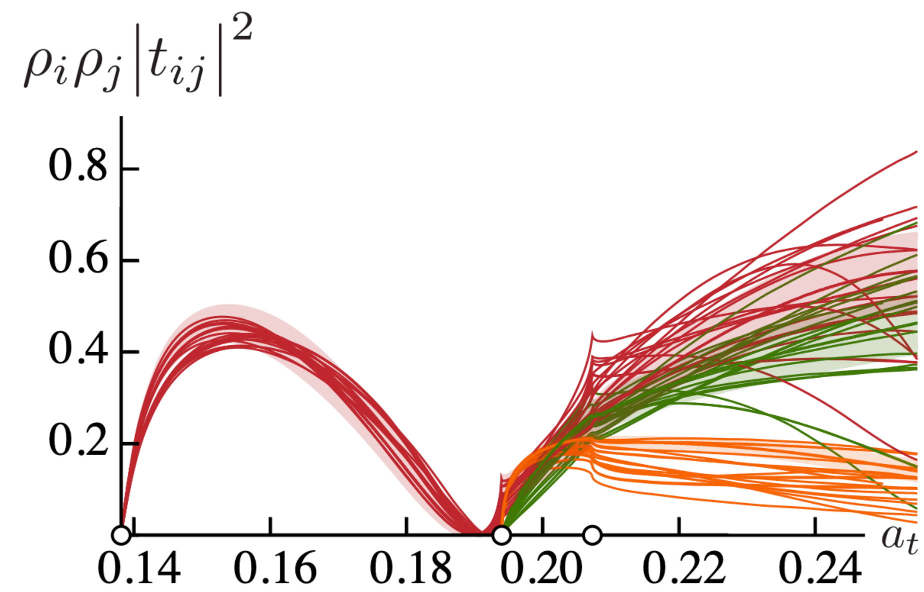
  extend studies involving an external current

  push more channels into the precision regime

Big Picture







A thriving field, with much more to come…

Thanks for listening!